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TTWO0213: Audit of Distributional Weight Analysis

Task 2: Review of DfT's Technical Paper

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ITS

Contents

1 Introduction

As part of the study 'TTWO0213 Audit of Distributional Weight Analysis' being undertaken by Arup and ITS Leeds for the Department for Transport, 'Task 2' involves reviewing a technical paper produced by DfT entitled 'Investigating distributional weighting in UK transport appraisal'.

This report forms part of that review – the other part being an annotated version of DfT's technical paper. The scope of this report is as follows:

- First, there is an exercise in replicating the derivation of the key formulae given in DfT's technical paper. The purpose of this exercise is to check and validate DfT's working.
- Second, once the key formulae have been derived, there is consideration of the manner in which DfT has implemented these formulae in the empirical work reported in the paper.
- Finally, whilst the above two parts of the report focus on the detail of DfT's formulae, the third part of the report considers the practical application of distributive weights in policy work. This exercise provokes some broader observations regarding DfT's formulae.

2 Replicating the derivation of DfT's key formulae

2.1 Basic definitions and concepts

Following the technical annex in DfT's paper, we begin by assuming the following utility function for individual (or group/segment) *i*:

$$U_i = \begin{cases} \dfrac{Y_i^{1-\mu}}{1-\mu} & \text{if } \mu \neq 1 \\ \log(y) & \text{if } \mu = 1 \end{cases} \quad (1)$$

where U_i is the utility to individual *i*, Y_i is individual *i*'s income and μ is a parameter (which will be elaborated upon below). This functional form, which in essence follows Layard et al. (2008), is recommended by the Green Book (HMT, 2022) in the context of distributional weighting, and gives rise to a convenient expression for the marginal utility of income (MUY):

$$MUY_i = \frac{\partial U_i}{\partial Y_i} = Y_i^{-\mu} \quad (2)$$

That is to say, MUY is expressed in 'elasticity' form, such that μ can be directly interpreted as the elasticity of the MUY. Since it is well established that MUY will reduce as income increases, the expectation is that μ>0. Indeed, based on research by Layard et al. (2008), HMT recommends μ=1.3.

Whereas the DfT paper couches utility/social welfare in absolute form, it is more natural in the context of economic appraisal – which is concerned with the *change* in utility and social welfare associated with the *change* in income stimulated by a given scheme – to couch utility/social welfare in differential form. In the case of a single individual, this would in general terms be given by:

$$dU_i = MUY_i dY \quad (3)$$

Then aggregating individual utilities *i*=1,…,*I* across society, the corresponding change in social welfare would be given by:

$$dU_s = \sum_{i=1}^{I} \Omega_i MUY_i \, dY \quad (4)$$

where the additional parameter Ω_i denotes the 'policy' weight imposed by government to adjust the influence of individual *i* within the SWF[1]. In these terms, it is generally understood that the **distributional weight** of individual *i* within the SWF is represented by the product of that individual's policy weight and his/her marginal utility, i.e.

$$DW_i = \Omega_i MUY_i \quad (5)$$

Since the MUY reduces as income increases, the implication follows that, if *all* individuals were in practice given equal weighting within the SWF (i.e. assigned the same DW), then richer individuals would in effect be given greater policy weight than poorer individuals. If policymakers wish to correct for this imbalance, then they need to apply a schedule of varying DWs which achieves the desired level of redistribution.

Against this background, it is useful to note the rationale for DWs as given in HMT's Green Book (p96):

"A3.3 When assessing costs and benefits of different options it may be necessary or desirable to 'weight' these costs and benefits, depending on which groups in society they fall on. This is in addition to estimating the 'unweighted' costs and benefits, which is the minimum requirement of Social CBA. In weighted analysis, financial benefits for lower income households are given a higher social value than the equivalent benefits for higher income households. Weighted estimates should be presented alongside unweighted estimates to demonstrate the impact of the weighting process.

[1] In this simple case we assume that all individuals receive the same change in income, but the analysis readily generalises to the case where different individuals receive different income changes.

A3.4 The basis for distributional weights is the economic principle of the diminishing marginal utility of income. It states that the value of an additional pound of income is higher for a low income recipient and lower for a high-income recipient. Broadly a value of 1 for the marginal utility of income would indicate that the utility of an additional pound is inversely proportional to the income of the recipient. An additional £1 of consumption received by someone earning £20,000 per year would be worth twice as much than to a person earning £40,000. Higher estimates of the marginal utility of income will mean the value of an additional pound declines more quickly relative to increases in income".

In essence, DfT's technical annex derives specific formulations for DWs using alternative numeraires. The subsequent sub-sections will repeat the same exercise for three such cases:

- Firstly, income-based DWs – which is perhaps the most conventional approach and that promoted by HMT.
- Secondly, income-based DWs, but now adjusting for the burden of tax.
- Thirdly, travel time-based DWs, adjusting for the burden of travel time.

2.2 Derivation of income-based DWs

Whilst the relevant commentary in DfT's paper is somewhat in conflict, the substantive working in the annex appears to commence (specifically A.5) from a situation where, **on average**, a given intervention generates £1 of (monetised) utility for each individual, thereby generating aggregate (monetised) welfare of *I* for society as a whole. Stating this in formal terms:

$$dU_{ave,£} = \frac{1}{MUY_s}\sum_{i=1}^{I}\frac{1}{I}MUY_i dY = 1 \quad (6)$$

$$\Rightarrow MUY_s = \frac{1}{I}\sum_{i=1}^{I}MUY_i\, dY \quad (7)$$

where MUY_s denotes the marginal utility of income of society and the subscript *ave* denotes 'average'.

Comparing (6) to (4), the division by *I* makes clear that (6) pertains to the **average** individual whereas (4) pertains to **society**. Another distinguishing feature is that (6) implicitly assumes that the policy weight $\Omega_i=1/MUY_s$ for all $i=1,\dots,I$. Three implications then follow:

a) Since the policy weight is given by the inverse of the MUY of society, it would at face value[2] seem that utility is expressed in money units rather than utils (hence the £ subscript).

[2] We will return to this point in section 3.

b) The policy weight is in effect constant across all individuals.

c) Given b), the policy weight is neutral and the aggregation is weighted simply by the MUY of each individual – this is consistent with Fujiwara (2010), which is the basis of GB guidance on distributional weights.

DfT's paper states that objective of their subsequent working is to *"solve for a reference level of income Y_{ref}, defined such that giving £1 to that person is worth £1 of social welfare"*. In terms of (6), this relates to the restricted case where $dY = 1$, i.e.

$$dU_{ref,£} = \frac{1}{MUY_s}\sum_{i=1}^{I} \frac{1}{I} MUY_i = 1 \quad (8)$$

$$= \frac{\overline{MUY}}{MUY_s} = 1 \quad (9)$$

$$\Rightarrow DW_{ref} = \frac{\overline{MUY}}{MUY_s} = 1 \quad (10)$$

Thus, in this reference case where the average MUY is equal to the social MUY, the implied DW is $\overline{MUY}/MUY_s = 1$, meaning that there is no need to redistribute to the representative individual with average marginal utility of income.

Again following DfT's paper, the working now proceeds to derive the level of income associated with the representative individual. Following the same steps as before for this case, the marginal utility of income will in this case be given by:

$$MUY_{ref} = Y_{ref}^{-\mu} \quad (11)$$

Where Y_{ref} is the reference income.

Then considering the case where a given intervention generates £1 of (monetised) utility for the representative individual, the relevant differential can in general terms be written:

$$dU_{ref,£} = \frac{MUY_{ref}}{MUY_s} = 1 \quad (12)$$

Substituting for MUY_{ref} from (11):

$$dU_{ref,£} = \frac{1}{MUY_s} Y_{ref}^{-\mu} = 1 \quad (13)$$

Further substituting for MUY_s from (7):

$$dU_{ref,£} = \frac{Y_{ref}^{-\mu}}{\frac{1}{I}\sum_{i=1}^{I} MUY_i} = 1 \quad (14)$$

Then also substituting for MUY_i from (1):

$$dU_{ref,£} = \frac{{Y_{ref}}^{-\mu}}{\frac{1}{I}\sum_{i=1}^{I} {Y_i}^{-\mu}} = 1 \quad (15)$$

$$\Rightarrow {Y_{ref}}^{-\mu} = \frac{1}{I}\sum_{i=1}^{I} {Y_i}^{-\mu} \quad (16)$$

$$\Rightarrow Y_{ref} = \left(\frac{1}{I}\sum_{i=1}^{I} {Y_i}^{-\mu}\right)^{-\frac{1}{\mu}} = \overline{MUY}^{-\frac{1}{\mu}} \quad (17)$$

Having considered the impacts of a given intervention on both individual *i* and the representative individual, it remains to combine both perspectives in the derivation of a schedule of distributive weights (DWs) – reflecting the degree of divergence in individual i's income from that of the representative individual.

Given the particular specification of the policy weight adopted here, the distributive weight of individual *i* is given by:

$$DW_i = \frac{MUY_i}{MUY_s} \quad (18)$$

Substituting for the MUYs of the individual and society, from (2) and (7) respectively:

$$DW_i = \frac{{Y_i}^{-\mu}}{\frac{1}{I}\sum_{i=1}^{I} {Y_i}^{-\mu}} \quad (19)$$

$$= \frac{{Y_i}^{-\mu}}{{Y_{ref}}^{-\mu}} \quad (20)$$

$$= \left(\frac{Y_{ref}}{Y_i}\right)^{\mu} \quad (21)$$

Comment: The DW for individual i is here derived as the ratio between the MUYs for individual i and the reference individual – which recovers the formulation given in DfT's paper. To interpret, remembering that the Green Book recommends μ=1.3, individuals whose income is less than the reference individual will be assigned a weight greater than one, and vice versa.

That said, a question which emerges is whether using different definitions of the marginal utility of income – respectively varying across individuals in terms of MUY_i and being constant in terms of MUY_S – is theoretically consistent, and in what units the resulting measure will be. It would seem that MUY_i converts money into individual utility, whereas MUY_S converts money into social utility. But can the inverse of MUY_S then be used to convert individual utility back into money – as claimed here?

2.3 Derivation of income-based DWs adjusted for tax

Having derived income-based DWs as above, DfT's paper then proceeds to repeat the same exercise but this time taking account of variation in the tax burden across individuals. In this case, DfT's paper gives the result but not the working. Therefore, we here attempt to reproduce the result from first principles.

Analogous to section 2.2, assume that the utility function for individual *i* is given by:

$$U_i = \frac{\left((1-t)\cdot Y_i\right)^{1-\mu}}{1-\mu} \quad (22)$$

where t is a constant proportional tax on income[3], and the corresponding marginal utility of *gross* income for individual *i* is given by:

$$MUY_i = \frac{(1-t)}{\left((1-t)\cdot Y_i\right)^{\mu}} \quad (23)$$

Alternatively, and following the approach taken in DfT's paper, marginal utility can be expressed in terms of *net* income:

$$\widetilde{MUY}_i = \tilde{Y}_i^{-\mu} \quad (24)$$

where $\tilde{Y}_i = (1-t)\cdot Y_i$.

Analogous to section 2.2, consider a situation where, **on average**, a given intervention generates £1 of (net monetised) utility for each individual, thereby generating aggregate (net monetised) welfare of *I* for society as a whole:

$$\widetilde{dU}_{ave,£} = \frac{1}{\widetilde{MUY}_s}\frac{\sum_{i=1}^{I} T_i\tilde{Y}_i^{-\mu}}{\sum_{i=1}^{I} T_i}d\tilde{Y} = 1 \quad (25)$$

$$\Rightarrow \widetilde{MUY}_s = \frac{\sum_{i=1}^{I} T_i\tilde{Y}_i^{-\mu}}{\sum_{i=1}^{I} T_i}d\tilde{Y} \quad (26)$$

where in this case (and following DfT's paper), average marginal utility is calculated as a weighted average over individuals, with each individual weighted by his/her tax burden $T_i = t\cdot Y_i$.

Analogous to section 2.2, now solve for a reference level of net income $\tilde{Y}_{ref}$, in which case (25) and (26) simplify to, respectively:

[3] Of course, this is a simplification of the UK income tax regime, which is designed to be progressive.

$$\widetilde{dU}_{ref,£} = \frac{1}{\widetilde{MUY}_s} \frac{\sum_{i=1}^{I} T_i \tilde{Y}_i^{-\mu}}{\sum_{i=1}^{I} T_i} = 1 \quad (27)$$

$$\Rightarrow \widetilde{MUY}_s = \frac{\sum_{i=1}^{I} T_i \tilde{Y}_i^{-\mu}}{\sum_{i=1}^{I} T_i} \quad (28)$$

Now define the marginal utility of income for a representative individual associated with the reference net income:

$$\widetilde{MUY}_{ref} = \tilde{Y}_{ref}^{-\mu} \quad (29)$$

In which case:

$$\widetilde{dU}_{ref,£} = \frac{Y_{ref}^{-\mu}}{\widetilde{MUY}_s} = 1 \quad (30)$$

Substituting for $\widetilde{MUY}_s$ from (28):

$$\widetilde{dU}_{ref,£} = \frac{\sum_{i=1}^{I} T_i}{\sum_{i=1}^{I} T_i \tilde{Y}_i^{-\mu}} \tilde{Y}_{ref}^{-\mu} = 1 \quad (31)$$

$$\Rightarrow \tilde{Y}_{ref}^{\mu} = \frac{\sum_{i=1}^{I} T_i}{\sum_{i=1}^{I} T_i \tilde{Y}_i^{-\mu}} \quad (32)$$

$$\Rightarrow \tilde{Y}_{ref} = \left(\frac{\sum_{i=1}^{I} T_i}{\sum_{i=1}^{I} T_i \tilde{Y}_i^{-\mu}} \right)^{\frac{1}{\mu}} \quad (33)$$

Given the particular form of the utility function adopted here, as well as the particular specification of the policy weight, the distributive weight of individual *i* is given by:

$$\widetilde{DW}_i = \frac{\widetilde{MUY}_i}{\widetilde{MUY}_{ref}} = \frac{\tilde{Y}_i^{-\mu}}{\tilde{Y}_{ref}^{-\mu}} = \left(\frac{\tilde{Y}_{ref}}{\tilde{Y}_i} \right)^{\mu} \quad (34)$$

It can be seen that, apart from the tax adjustment on income, (34) corresponds to (21).

Comment: This adjustment to the GB income-based distributive weight would not seem unreasonable, but it affects the definition of income and the averaging process rather than the mathematical formulae per se.

2.4 Combining the DWs from sections 2.2 and 2.3

Point A.14 of DfT's technical annex asserts that: *"…the benefits (or appraisal valuations such as average VTTS) expressed in terms of one 'unit of account' can easily be converted to any other unit of account, by multiplying them by the reference incomes to the power μ".*

Point A.13 demonstrates this, comparing the income-based DW and income-based DW adjusted for tax from sections 2 and 3 above. On this basis, let us derive:

$$\frac{\widetilde{DW}_i}{DW_i} = \frac{\tilde{Y}_{ref}{}^{\mu}}{\tilde{Y}_i{}^{\mu}} \Big/ \frac{Y_{ref}{}^{\mu}}{Y_i{}^{\mu}} = \left(\frac{(1-t)\cdot Y_{ref}}{(1-t)\cdot Y_i}\right)^{\mu} \Big/ \left(\frac{Y_{ref}}{Y_i}\right)^{\mu} = 1 \quad (35)$$

Whilst DfT's assertion holds, there is no need for conversion in the case considered here, since the MUY is common to the two units of account.

Comment: Whilst the mathematics of this result are not disputed, later discussion in Section 4 will question whether it is really necessary to employ more than one version of DWs – in which case there would be no need to convert from one 'unit of account' to another.

2.5 Derivation of travel time-based DWs

Having derived income-based DWs both with and without adjustment for the tax burden, DfT's paper then proceeds to translate the concept of DWs to changes in travel time rather than changes in income. Again, the DfT paper gives the result but not the working. Therefore, we here attempt to reproduce the result from first principles, following the same steps as before. The distinguishing feature here is that the DW is derived in terms of a marginal change in travel time rather than a marginal change in income. That said, the reference individual could in principle be defined in either of two ways – in terms of income or travel time.

2.5.1 Defining the reference individual in terms of income

In effect[4], DfT's paper **asserts** that the average (monetised) value of travel time savings generated by a transport intervention is given by the following identity:

$$dU_{ave,£} = \frac{1}{MUY_s} \frac{\sum_{i=1}^{I} \tau_i MUY_i VTTS_i}{\sum_{i=1}^{I} \tau_i} d\tau \quad (36)$$

where in this case, a weighted average is taken of the product $MUY_i \cdot VTTS_i$ over individuals, with each individual weighted by his/her travel time (or distance as a proxy) τ_i.

It should be stressed that DfT's paper asserts but does not derive (36) from first principles, nor does the paper make any comment as to the provenance of (36). Later discussion in Section 4 will reflect further on the basis of this identity, but for the moment we will proceed with (36) and derive a number of results which follow.

Note in passing that the following identity holds by definition:

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[4] As noted earlier, DfT's paper does not actually state utilities in differential form, but this is the most natural formulation in the context of scheme appraisal.

$$MUY_i VTTS_i = MUY_i \frac{MU\tau_i}{MUY_i} = MU\tau_i \quad (37)$$

Where $MU\tau_i$ denotes the marginal utility of travel time. On this basis, (36) could be written more succinctly as:

$$dU_{ave,£} = \frac{\overline{MU\tau}}{MUY_s} d\tau \quad (38)$$

where $\overline{MU\tau}/MUY_s$ is effectively an average distributionally-weighted VTTS. On the basis of (36), $\overline{MU\tau}$ could also be interpreted as a travel time-weighted average marginal utility of travel time, and it becomes apparent why DfT's concept of an 'equity VTTS' requires a constant marginal utility of travel time across all individuals (i.e. $MU\tau_i = MU\tau$ for all i=1,…,I).

Returning to (36) but substituting for MUY_i from (2) and MUY_S from (11):

$$dU_{ave,£} = \frac{1}{Y_{ref}{}^{-\mu}} \frac{\sum_{i=1}^{I} \tau_i Y_i{}^{-\mu} VTTS_i}{\sum_{i=1}^{I} \tau_i} d\tau \quad (39)$$

$$= \frac{\sum_{i=1}^{I} \tau_i DW_i VTTS_i}{\sum_{i=1}^{I} \tau_i} d\tau \quad (40)$$

Finally, if we restrict (40) to the case where each and every individual/traveller saves 1 minute, then $d\tau$=1, and we derive the average change in utility expressed in VTTS units (£/min) as opposed to money units.

$$dU_{ave,£/min} = \frac{\sum_{i=1}^{I} \tau_i DW_i VTTS_i}{\sum_{i=1}^{I} \tau_i} = VTTS_{DW} \quad (41)$$

This recovers the equation given in section 4.1 of DfT's paper, which they refer to as a 'distributionall-weighted VTTS'.

Comment: Since (41) is the key formula taken forward by DfT in their empirical work, it is noted that the above working successfully derives this formula from the starting point of (36) – but it should be reiterated that (36) is itself asserted rather than derived. This point will be considered further in Section 4.

2.5.2 Defining the reference individual in terms of travel time

An alternative way of proceeding would be to define the representative individual in terms of travel time rather than income, such that the policy weight is now given by $\Omega_i = 1/MU\tau_s$ for all i=1,…,I. On this basis, consider a situation where, **on average**, a given transport infrastructure intervention generates welfare (now quantified in travel time units) of 1 minute for each individual, thereby generating aggregate welfare of I minutes for society as a whole:

$$dU_{ave,\tau} = \frac{1}{MU\tau_s} \frac{\sum_{i=1}^{I} \tau_i MUY_i VTTS_i}{\sum_i^n \tau_i} d\tau = 1 \quad (42)$$

Now solve for a reference level of travel time by assuming $d\tau = 1$ at the outset (i.e. a 1 minute time saving), in which case (42) simplifies to:

$$dU_{ref,\tau} = \frac{1}{MU\tau_s}\frac{\sum_{i=1}^{I}\tau_i MUY_i VTTS_i}{\sum_{i=1}^{n}\tau_i} = 1 \quad (43)$$

$$\Rightarrow MU\tau_s = \frac{\sum_{i=1}^{I}\tau_i MUY_i VTTS_i}{\sum_{i=1}^{I}\tau_i} = \overline{MU\tau} \quad (44)$$

Substituting for MUY_i from (2):

$$MU\tau_s = \frac{\sum_{i=1}^{I}\tau_i Y_i^{-\mu} VTTS_i}{\sum_{i=1}^{I}\tau_i} = \overline{MU\tau} \quad (45)$$

Now repeating (43) from the specific perspective of a representative individual with both reference income *and* reference VTTS[5]:

$$dU_{ref,\tau} = \frac{Y_{ref}^{-\mu} VTTS_{ref}}{MU\tau_s} = 1 \quad (46)$$

$$\Rightarrow MU\tau_s = Y_{ref}^{-\mu} VTTS_{ref} \quad (47)$$

Given the particular specification of the policy weight adopted here, the distributive weight of individual *i* is given by:

$$DW_{i,\tau} = \frac{MUY_i VTTS_i}{MU\tau_s} = \frac{MU\tau_i}{MU\tau_s} \quad (48)$$

$$= \left(\frac{Y_i}{Y_{ref,\tau}}\right)^{-\mu} \frac{VTTS_i}{VTTS_{ref}} \quad (49)$$

In this way, the DW for individual i is now derived as the product of two components:

i) an income-based DW but with the reference income now a weighted average with travel time (or distance) as the weight, and;

ii) a VTTS-based DW.

Remembering that the Green Book recommends μ=1.3, (49) implies that whereas individuals with incomes less than the reference income will receive an income-based DW greater than

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[5] Remembering from (37) that the product of the marginal utility of income and the VTTS is the marginal utility of travel time. It should be acknowledged that definition of the reference travel time is open to debate – but in the derivation here it is implicitly assumed that this is given by the travel time of the individual with reference income (and the VTTS of that individual then provides the basis for converting between time and money).

one, this will be offset to a greater or lesser extent by the VTTS-based component – since those on lower incomes would be expected to have lower VTTS. Whether the VTTS-based DW will be less than or greater than one will be an empirical issue.

Comment: Further elaboration will be given on the various influences of income when discussing DfT's empirical work in Section 3 below.

3 Interpretation of DfT's empirical work

In essence, the empirical work presented in section 4 of DfT's paper is focussed upon the implementation of equation (41). The empirical work reported by DfT compares several variants of (41) in terms of:

- Calculating the income of the reference individual via (17), or instead adopting the Green Book definition of the representative individual (as the individual with median income);
- Employing alternative definitions of income: equivalised vs. unequivalised; disposable income vs. household consumption; household consumption with vs. without housing costs.
- Adjusting for the tax burden via (34), or not adjusting for tax via (21).

3.1 The role of income

Whilst the impact of the above variants on the resulting DWs is of interest, the overarching question considered in DfT's paper is whether and to what extent each of the substantive components of (41) is correlated with income – since this informs understanding of the distributional implications of (41) in totality.

In an attempt to shed light on this question, DfT's paper considers (41) in totality, as well as two reduced forms of (41):

$$\overline{VTTS} = \frac{\sum_i^n \tau_i VTTS_i}{\sum_{i=1}^n \tau_i} \qquad (50)$$

$$\overline{DW} = \frac{1}{{Y_{ref}}^{-\mu}} \frac{\sum_{i=1}^n \tau_i {Y_i}^{-\mu}}{\sum_{i=1}^n \tau_i} \frac{\sum_{i=1}^n \tau_i \left(\frac{Y_i}{Y_{ref}}\right)^{-\mu}}{\sum_{i=1}^n \tau_i} \qquad (51)$$

where (50) is the standard equity VTTS currently used in TAG for non-work journey purposes, and (51) is the weighted-average of the distributional weights, with distance as the weight.

Considering the influence of income on each of the substantive components of (41), intuition suggests that:

a) The travel burden of individual i, τ_i, is expected to **increase** as his/her income increases;

b) The behavioural value of travel time savings for individual i, $VTTS_i$, is expected to **increase** as his/her income increases;

c) Given the formulation adopted here, the income-based distributive weight of individual i, DW_i, is expected to **decrease** as his/her income increases.

Furthermore, if the reference income is adjusted for the tax burden via (33), then $VTTS_{DW}$ is revised to:

$$\widetilde{dU}_{ave,£,\tau=1} = VTTS_{DW} = \frac{1}{\frac{\sum_{i=1}^{n} T_i \tilde{Y}_i^{-\mu}}{\sum_{i}^{n} T_i}} \frac{\sum_{i=1}^{n} \tau_i \tilde{Y}_i^{-\mu} VTTS_i}{\sum_{i=1}^{n} \tau_i} \quad (52)$$

In which case, a fourth substantive component is introduced, whereby:

d) The tax burden of individual i, T_i, is expected to **increase** as his/her income increases;

The overall impact of income on $VTTS_{DW}$ will be the net of impacts a), b), c) and d).

Note in passing that the alternative travel time-based DW (49) would in effect capture impacts a), b) and c), whilst the income-based DW (21) would in itself capture c) only.

DfT's empirical work proceeds by first generating $VTTS_{DWi}$, $VTTS_i$ and DW_i for each individual in the NTS sample used in DfT's 2014/15 national VTTS study (or NRTS equivalent), but focussing on two variants of (41), namely:

i) Based on the representative (i.e. median income) individual as per the Green Book, here applied in terms of equivalised disposable income.

ii) Based on the representative individual as per (17), here applied in terms of equivalised consumption with adjustment for the tax burden.

Two analyses are then undertaken; firstly, the averages (41), (50) and (51) are generated across the sample for a range of different segmentations (by mode, purpose and region) for both variants i) and ii) above; secondly, regressions are undertaken on the individual-level data (i.e. before averaging) for variant i) only, to understand the relationship between distributionally-weighted and equity-weighted VTTS and income (and correlates of income, namely travel cost, travel time and distance).

The results from DfT's regressions are reproduced below (Tables 3 and 4 in their paper); since the models were specified in log-linear form, the reported parameters can be directly interpreted as elasticities. With reference to the 'first best' models, it can be seen that the introduction of GB distributive weights has a marked impact on the income elasticity (so much so that the sign switches from positive to negative), whilst the cost and distance elasticities remain reasonably unchanged. That is to say, the introductive of DWs has the effect of making time saving benefits more progressive.

Whilst it is tempting to conclude from the two regressions that $VTTS_b$ has a regressive effect (since the income elasticity is positive) whilst $VTTS_{DW}$ has a progressive effect (income elasticity is negative), one cannot conclude this definitively without better understanding any correlation between the income elasticity and the travel time and cost elasticities.

	First best models		**Second best models**		**Third best models**	
Text	**VTTS_b**	**VTTS_dw**	**VTTS_b**	**VTTS_dw**	**VTTS_b**	**VTTS_dw**
Time	-0.60	-0.55	-0.68	-0.52	-	-
Cost	0.63	0.55	0.77	0.50	-	-
Income	0.60	-0.20	-	-	-	-
Distance	-	-	-	-	0.26	0.12

Table 1 Car Commute

	First best models		**Second best models**		**Third best models**	
Text	**VTTS_b**	**VTTS_dw**	**VTTS_b**	**VTTS_dw**	**VTTS_b**	**VTTS_dw**
Time	-0.29	-0.28	-0.31	-0.24	-	-
Cost	0.65	0.63	0.70	0.52	-	-
Income	0.30	-0.63	-	-	-	-
Distance	-	-	-	-	0.46	0.32

Table 2 Rail Commute

3.2 The econometric specification

Now examining the econometric work in more detail, DfT's models basically involve regressing the construct $DW_i \cdot VTTS_i$ across the *I* individuals within NTS/NRTS samples (i.e. before any form of weighting/averaging) against income and travel time, cost and distance. It should be highlighted that all of the components of $DW_i \cdot VTTS_i$ are deterministic and not associated with error. That is to say, given the observed income of individual *i*, a distributional weight is calculated using the income-based formula (21). Similarly, for each individual *i* in the sample, the corresponding behavioural VTTS is calculated given a pre-determined formula (55) from the 2014/15 national VTTS study.

Against this background, note that the following identity holds by definition:

$$\ln(DW_i \cdot VTTS_i) = \ln(DW_i) + \ln(VTTS_i) \qquad (53)$$

where, following (21), the first part can conveniently be transformed into a linear additive form:

$$\ln(DW_i) = \mu \cdot (\ln(Y_{med}) - \ln(Y_i)) \qquad (54)$$

With reference to Hess et al. (2017), the behavioural VTTS from the 2014/15 study is given by:

$$VTTS_i = \exp(\kappa_{SP1} \cdot a) \frac{\exp(\kappa_{SP1} \cdot b) - 1}{\kappa_{SP1} \cdot b} \zeta_{SP1} \prod_m z_{mi}^{\lambda m} \prod_n \zeta_n^{z_{ni}} |\Delta t|^{\kappa_{SP1}-1} \qquad (55)$$

which due to its multiplicative form also transforms into a linear additive form:

$$\ln(VTTS_i) = K + \sum_m \lambda_m \cdot \ln(z_{mi}) + \sum_n z_{ni} \cdot \ln(\zeta_n) \qquad (56)$$

It is appropriate to make some comments on the various terms above:

- $\exp(\kappa_{SP1} \cdot a) \frac{\exp(\kappa_{SP1} \cdot b)-1}{\kappa_{SP1} \cdot b} \zeta_{SP1}$ and $|\Delta t|^{\kappa_{SP1}-1}$do not vary across respondents and can therefore be treated as constants, here denoted *K*;
- z_{mi} refers to a set of continuous variables (such as income, time, cost and distance) with elasticities λ_m;
- z_{ni} refers to a set of 0/1 dummy variables with transformed parameters $\ln(\zeta_n)$.

The term *K* can be further extended by including $\mu \ln(Y_{med})$, which is also a constant, such that:

$$\ln(DW_i \cdot VTTS_i) = K - \mu \cdot \ln(Y_i) + \sum_m \lambda_m \cdot \ln(z_{mi}) + \sum_n z_{ni} \cdot \ln(\zeta_n) \qquad (57)$$

Since the influence of income will be captured by both μ and λ_m, the income elasticity in the subsequent regression will be a mixture of these two effects.

In effect, DfT's econometric work involves estimating an ex post regression, whereby forecasts of $\ln(DW_i \cdot VTTS_i)$ for each individual *i* within the NTS/NRTS generated by (57) are regressed against income and travel cost, time and distance for that individual also from NTS/NRTS. More specifically, DfT estimated the following model:

$$\ln(DW_i \cdot VTTS_i) = \alpha + \beta_I \ln(Y_i) + \beta_T \cdot \ln(T_i) + \beta_C \cdot \ln(C_i) + \epsilon_i \qquad (58)$$

Where *T* is travel time and *C* is travel cost.

The key question to consider is whether (58) is a faithful reproduction of (57) – and this provokes a number of comments.

First, it should be acknowledged that the definition of income which underpins the parameters of the behavioural model (57), i.e. based on the SP sample, may not necessarily be consistent with that which underpins the regression model (58), i.e. based on NTS/NRTS.

Comment: The income measure used in the DWs should be fully aligned with the income measure used in the VTTS – otherwise there is a risk of misspecification.

Second, in comparison to (57), a reduced set of variables is included in the estimated regression model (58) and the error term will therefore **at least** include:

$$\epsilon_i = \sum_n z_{ni} \cdot \ln(\zeta_n) \qquad (59)$$

This error will also include additional continuous variables – such as distance, included within the behavioural VTTS models but not the regressions – and additional measurement error due to the use of inconsistent income measures between DW_i and $VTTS_i$. Since these missing variables are non-random, the estimated time, cost, distance and income elasticities will differ from those in the original behavioural VTTS models.

Comment: Missing variables and measurement error in the included variables are likely to generate bias in the parameter estimates of the regression model. Therefore, income, time and cost elasticities may be higher or lower than those estimated in the 2014/15 models – but this is entirely the result of model error.

In other words, the estimated parameters are likely to be **biased** and do not resemble the true parameters, either originating from DW_i or from $VTTS_i$. Since the true data generating process is unrelated to the distance and NTS expansion weights (the latter are only used for generating an average VTTS), unweighted linear regression models for disaggregate should in principle be capable of retrieving the true parameters (i.e. those estimated in the 2014/15 study).

Comment: If missing variables are correlated with the distance and NTS expansion weights, then this may induce further bias in the parameters already estimated using weighted linear regressions. We recommend that the models are re-estimated as unweighted regressions.

Third, DfT's models either include additional variables related to geography in (58) (i.e. the weighted linear regressions), or relate the unexplained error in VTTS to geography. In the majority of the behavioural models from the 2014/15 VTTS study, geography is not included as an explanatory variable. Hence, by default, differences in VTTS across regions are not expected. Any such geographical variation is either due to missing variables, inconsistency in income measures between DW_i and $VTTS_i$, and/or the use of weighted linear regression as opposed to unweighted linear regression – all of which will induce bias or unexplained variation. In other words, there is a risk of incorrectly attributing model misspecification to geographical variation.

Comment: When all explanatory variables are included in the models, there should by default be no variation in VTTS across geographic regions – unless geography is an explanatory variable in the 2014/15 behavioural models. On this basis, the geographical variation claimed in DfT's paper is mainly the result of misspecification in the regressions.

Since the functional form of $DW_i \cdot VTTS_i$ is known, including the relevant elasticities with respect to income, time, cost etc., the results of interest are obtainable without estimation on the NTS and NRTS data. The behavioural VTTS models can be re-estimated using income measures that are consistent with those used in DW_i (see Tijong et al., 2022), in which case the income elasticity of $DW_i \cdot VTTS_i$ will be to $\lambda_{inc} - \mu$, where the former captures the increase in the VTTS due to income and the latter the reduction in the distributional weight of those having higher income levels.

Comment: Our recommendation is to re-estimate the 2014/15 models using a measure of income which is common to DW_i and $VTTS_i$ – the parameters of interest can then be obtained and directly applied.

4 Application of DWs in practice – and implications for DfT's formulation

To recap on the key findings of this report thus far:

- Section 2 successfully reproduced DfT's formula for an income-based DW (21), but noted that the subsequent translation to a VTTS-based DW (36) is asserted by DfT rather than derived.
- Section 3 reviewed DfT's empirical application of (36), highlighting that income will influence this travel time-based DW through four distinct mechanisms – making the role of income complex and difficult to unpack. Furthermore, various technical concerns were noted around the specification of the econometric model – which lead us to question the inferences being drawn.

Having focussed thus far on the detail of DfT's formulae, this section of the report will now adopt a broader perspective, by considering the basic principles of applying DWs in policy work. This exercise will provoke some further observations on the travel time-based DW (36).

4.1 Appraising a standard transport investment project

For purposes of illustration, consider a transport scheme that generates travel time benefits for the population of Leeds, where individual *l* (i.e. *l* for Leeds) benefits by Δt_l minutes. In accordance with current TAG guidance, the monetised benefits of the scheme for a given year[6] would be given by:

$$B = \sum_{l=1}^{L} \overline{VTTS} \cdot \Delta t_l \qquad (60)$$

[6] For simplicity, benefits are here considered for a single year – but this does not obviate the need to consider the implications of distributional weighting for discounting.

Where L is the population of Leeds and $\overline{VTTS}$ is TAGs official 'equity' VTTS (i.e. equation (50) above).

If we wish to account for differences in the marginal utility of income across the population of Leeds, then we can express the social welfare implications of these travel time benefits in terms of the following social welfare function:

$$W = \sum_{l=1}^{L} DW_l \cdot \overline{VTTS} \cdot \Delta t_l \qquad (61)$$

According to textbook definitions (e.g. Johansson, 1987), DW_l would conventionally be defined in terms of the marginal utility of income of individual *i*, such that *W* as a whole would be defined in utility units or 'utils'.

In the context of distributional weighting, the Green Book adopts a somewhat different approach of defining DW_l as the ratio of the marginal utilities of income of individual *l* and the median income earner *med*. Substituting for DW_l on this basis, (61) can be restated:

$$W = \sum_{l=1}^{L} \frac{MUY_l}{MUY_{med}} \cdot \overline{VTTS} \cdot \Delta t_l \qquad (62)$$

Bearing in mind that MUY_{med} is a constant, it is debatable whether the SWF is now defined in units of monetised benefits, or remains in utils (albeit rescaled via the division by a constant).

According to economic theory, the SWF in effect 'cardinalises' the **individual** utilities U_i through three distinct operations, namely: quantifying individuals' marginal utilities, multiplying those marginal utilities by the relevant change in income (or in this case travel time), and then summing over individuals. However, strictly speaking, the resulting welfare measure *W* remains ordinal I– and is therefore applicable only for ranking schemes rather than determining their respective NSBs in absolute terms.

Against this background, the GB method as applied above would seem to imply three strong assertions:

1) That dividing the individual's utility benefits of time savings $MUY_l \cdot \overline{VTTS} \cdot \Delta t_l$ by MUY_{med} converts from utility units to income units.

2) That the resulting measure of social welfare is interpretable as a measure of monetised benefits for direct application in CBA.

3) That the resulting measure of social welfare is interpretable in cardinal terms, e.g. in the sense that the monetised benefits (as calculated above) can be offset against the costs to arrive at a measure of NSB for a scheme.

Comment: On the basis of points 1) to 3), it becomes apparent that the selection of MUY_{med} as the 'conversion factor' from utils to income is far from innocuous – the choice of factor will not *affect the relative distributional weights of different individuals, but it will affect the absolute measure of benefits which feeds into CBA.*

Notwithstanding these points, note the key property of (62) that, for a given average VTTS, since the MUY of individual *i* is expected to decrease with income, whilst the MUY of the median income earner is constant, time savings accruing to poorer groups in the population will be weighted more than those accruing to richer groups. This is the precise objective of distributive weighting in this context.

4.2 Including additional project benefits

Now consider that the transport scheme in Leeds delivers not only travel time savings but also improvements in reliability. It is straightforward to introduce additional project benefits such as reliability to the above framework, in which case (60) and (61) would be extended to, respectively:

$$B = \sum_{l=1}^{L} \overline{VTTS} \cdot \Delta t_l + \overline{VTTR} \cdot \Delta r_l \qquad (63)$$

$$W = \sum_{l=1}^{L} DW_l \cdot (\overline{VTTS} \cdot \Delta t_l + \overline{VTTR} \cdot \Delta r_l) \qquad (64)$$

where Δr_l is the change in travel time reliability from the scheme and $\overline{VTTR}$ is the TAG value for such improvements. The point made here is that distributional weights are best applied to the overall benefits (and costs) of the project as opposed to individual impacts. Having one approach for distributionally weighting VTTS and another approach for user costs, for example, could run the risk of applying different weights to different impacts.

Comment: At what level does DfT envisage applying distributional weights? At the level of time savings, or at the level of the transport project or scheme? The latter would encourage a more consistent approach to DWs.

4.3 A specific VTTS for the affected population

The 2014/15 UK national VTTS study deployed SP methods on a large sample of travellers in order to derive a functional form for the behavioural VTTS of non-work journeys. This demonstrated that the behavioural VTTS varies across individuals according to the general[7] functional relationship $VTTS_i = f(X_i; \beta)$, where X is a set of explanatory variables pertaining to individual *i* and β a set of associated parameters.

In the next step of translating behavioural values into appraisal values, the 2014/15 study derived a nationally representative average VTTS for the UK travelling population, by applying

[7] The more detailed functional relationship is given by equation (56).

the above functional relationship to each individual/journey recorded in the NTS to estimate the behavioural VTTS for that individual, and then taking a weighted average across individuals. In equation (65) below, α_i refers to the weight applied to individual *i* in the NTS so as to make this sample representative of the travelling population. To be precise, this weight includes both distance *and* expansion weights.

$$\overline{VTTS} = \frac{\sum_{i=1}^{I} \alpha_i \cdot VTTS_i}{\sum_{i=1}^{I} \alpha_i} \qquad (65)$$

We specifically distinguish between indexes *i* and *l,* where the former relates to individuals in the NTS sample used to calculate the nationally average VTTS, and the latter relates to the population of Leeds affected by the transport scheme in our illustration.

Of course, if we were able to observe all relevant characteristics of the population of Leeds, then we would be able to calculate $VTTS_l$ for each individual, as well as aggregate across individuals to calculate the overall scheme benefits and associated social welfare), respectively:

$$B = \sum_{l=1}^{L} \Delta t_l \cdot VTTS_l \qquad (66)$$

$$W = \sum_{l=1}^{L} DW_l \cdot \Delta t_l \cdot VTTS_l \qquad (67)$$

The distinguishing feature of (66) and (67) relative to their earlier counterparts (60) and (61) is that, given quantification of an individual-specific VTTS for all individuals in Leeds, there is no longer a need for a representative average, and as a result the weights α_i are also unnecessary. In this context, the $VTTS_l$ monetises the travel time benefits to individual *l* and DW_l rescales these benefits on the basis of distributional considerations.

Comment: In practice, are we likely to be able to identify a VTTS and DW for each and every individual affected by a scheme? Probably not.

4.4 The proposed equation

Presumably for the above reason, DfT's paper proposes an *average* VTTS, of the form:

$$\overline{VTTS}_{DW} = \frac{\sum_{i=1}^{I} \alpha_n \cdot DW_i \cdot VTTS_i}{\sum_{i=1}^{I} \alpha_i} \qquad (68)$$

The implication here is that the distributional weight is applied at the NTS (or NRTS for rail) sample level (or some segment thereof) already corrected for representativeness of the UK travelling population – as opposed to the population benefitting from the transport scheme.

If one were to apply $\overline{VTTS}_{DW}$ to the population of Leeds, then one would obtain

$$B = \sum_{l=1}^{L} \Delta t_l \cdot \overline{VTTS}_{DW} \qquad (69)$$

As a result the benefits would, in comparison to (68), be uprated or downrated by the ratio $\frac{\overline{VTTS}_{DW}}{\overline{VTTS}}$, but $\overline{VTTS}_{DW}$ will be a standard value applied to each individual across the population of Leeds (or segments thereof).

Comments:

- *What is the motivation for applying DW_i to the NTS (or NRTS) sample corrected for representativeness – and not the scheme population?*
- *Should DW_i be treated as an averaging weight, or is it an inflator of $VTTS_i$? If the former, then DW_i should also be included in the denominator.*
- *In order to have the desired policy effect, should additional distributional weights be introduced at the scheme level?*

5 Summary and conclusions

This report has presented a detailed review of a technical paper produced by DfT entitled 'Investigating distributional weighting in UK transport appraisal'. The review comprised three parts:

- First, there was an exercise in replicating the derivation of the key formulae given in DfT's technical paper. The purpose of this exercise was to check and validate DfT's working.

  We successfully reproduced DfT's formula for an income-based DW (21), but noted that the subsequent translation to a VTTS-based DW (36) was asserted by DfT rather than derived.

- Second, once the key formulae had been derived, there was consideration of the manner in which DfT had implemented these formulae in their empirical work.

  We reviewed DfT's empirical application of the VTTS-based DW (36), highlighting that income will influence this DW through four distinct mechanisms – making the role of income complex and difficult to unpack. Furthermore, various technical concerns were noted around the specification of the econometric model – which led us to question the inferences being drawn by DfT.

- Third, there was discussion of the practical application of distributive weights in policy work.

DfT's paper stops short of such a discussion – which perhaps represents an omission, since the discussion exposed a number of substantive issues, in terms of the detail of the DW formulae, as well as the applicability of DWs to modelling and appraisal.

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