Household networks and bubbling: insights from percolation theory

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Summary

- Bubbling where any households join into groups of 2 or 3 substantially increases connectivity of the household network, with subsequent R_t likely to be above 1.
- Bubbling where households of **size 1** join another household of **size 1** have a small impact on network connectivity with, R_t remaining below 1.
- Bubbling where household of **size 1** join **any other household** have a relatively small impact on network connectivity, with R_t remaining below 1, *unless it is already close to 1*.
- Bubbling where households of size 2 join any other household increase network connectivity and could result in R_t above 1 if it is currently above 0.8.

Methods

We simulated contact networks of households, based on the distribution of household sizes from the 2011 census (Figure 1).

We assumed that all individuals in a household are fully connected. To capture between household transmission, we added one external link per person, linked to another person, chosen at random – we call this the *baseline network* with no bubbles. When each person is linked to a single other person outside their household, 92% of households are connected to each other. That is, there is a **giant component** and 92% of households are in the giant component.

We then remove a specific proportion, $(1-p)$, of links from the network at random. As $(1-p)$ increases, between-household links are removed from the baseline network and the giant component decreases in size. At the critical point, p_c, the network fragments abruptly and the giant component disappears - this is the **phase transition**.

In order to investigate when the giant component breaks up and the phase transition occurs, we simulated removing links from the baseline network and measured the number of households in the giant component and the average size of the other components in the network (also known as the **order parameter**). A discontinuity (peak) in the profile of the order parameter indicates the location of the phase transition.

We repeated this procedure for each network following different bubbling scenarios, estimating the location of the threshold for each case. Since each realization of the network is stochastic, we repeated this 10 times, for each scenario, and assessed the effects of random noise.

Estimating the location of the threshold relative to the current situation

At the point of phase transition, p_c, the between household reproduction number is close to one, $R_t \sim 1$. We assume that the reproduction number scales linearly with the probability of a link existing, $R_t \propto p$, with $p = 0 \equiv R_t = 0$. We then use this assumption to estimate the location of the current situation which we consider to be the baseline network, with estimates of the current reproduction number lying between $0.6<R_t<0.9$. See the shaded region in Figure 2.

Comparing that to location of the threshold for the other strategies, indicates whether they are below or above the critical threshold.

Simulating the effect of bubbling
We modelled bubbling by combining households of various sizes into larger ones, thus increasing the number of links those households form. We make the worst-case assumption that all households form bubbles.

We considered the following strategies:
2-bubbles: Two households join together at random.
3-bubbles: Three households join together at random.
1+1: Two households of size 1 join together to make a single household of size 2.
1+n: Households of size 1 join together with another household of random size.
2+n: Households of size 1 or 2 join with another household of random size.

For each bubbling strategy, we measured the proportion of households in the giant component and the order parameter for each proportion of removed links *p*. By varying *p*, we estimated the location of the critical threshold, $\boldsymbol{p_c}$, and assessed where it is in relation to the current situation.

If $\boldsymbol{p_c}$ is above (to the right of) the shaded region in Figure 2, we conclude that the strategy is **unlikely** to push R_t above 1; if it is below, we conclude it is **likely** to push R_t above 1. If $\boldsymbol{p_c}$ falls in the shaded region, the bubbling strategy could push R_t above 1, if it is already close to 1.

Results

- In 2-bubbling and 3-bubbling the phase transition point $\boldsymbol{p_c}$ is substantially lower than for the baseline network. This implies that should these be implemented, R_t would exceed 1.
- Similarly, bubbling households of size 2 with other households (strategy **2+n**), also appears to substantially reduce $\boldsymbol{p_c}$, and implies that R_t would also exceed 1.
- Bubbling strategies **1+1** and **1+n** have a minimal impact on the location of the phase transition compared to baseline with no bubbles, therefore have a minimal impact on the epidemic threshold. However, should the current estimate of R_t be close to 1, bubbling scenario **1+n** would also push R_t above 1.

Table 1: Comparison of bubbling strategies
The colours indicate whether the strategy is likely to push R_t above 1: (no, yes, maybe)

	Percolation threshold, p_c	**% of households in giant component with 1 outside link per person**	**% of households in giant component with 0.5 outside links per person**
Baseline	0.48	92%	3%
2-bubbles	0.23	100%	89%
3-bubbles	0.15	100%	97%
1+1	0.44	100%	26%
1+n	0.41	97%	41%
2+n	0.33	100%	80%

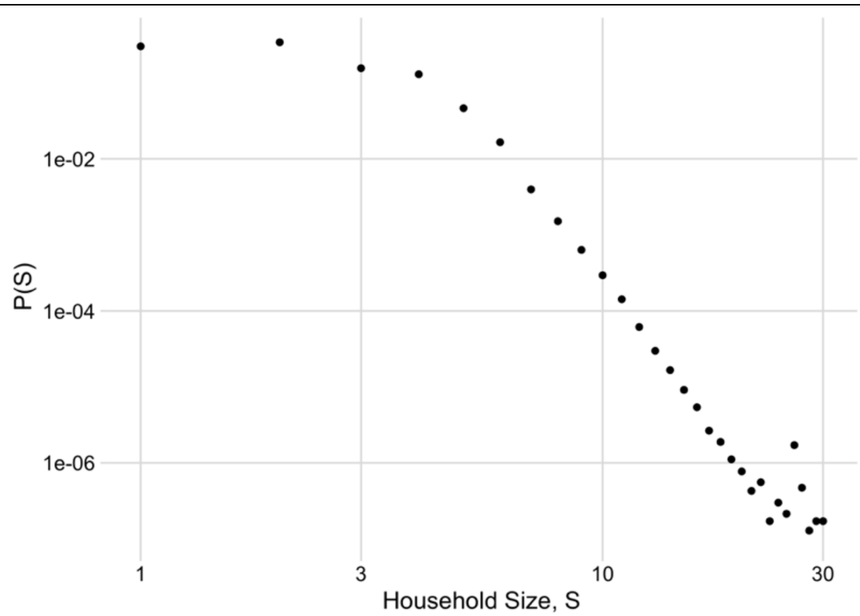

Figure 1: Distribution of household sizes in the UK from the 2011 ONS census (~23 million households).

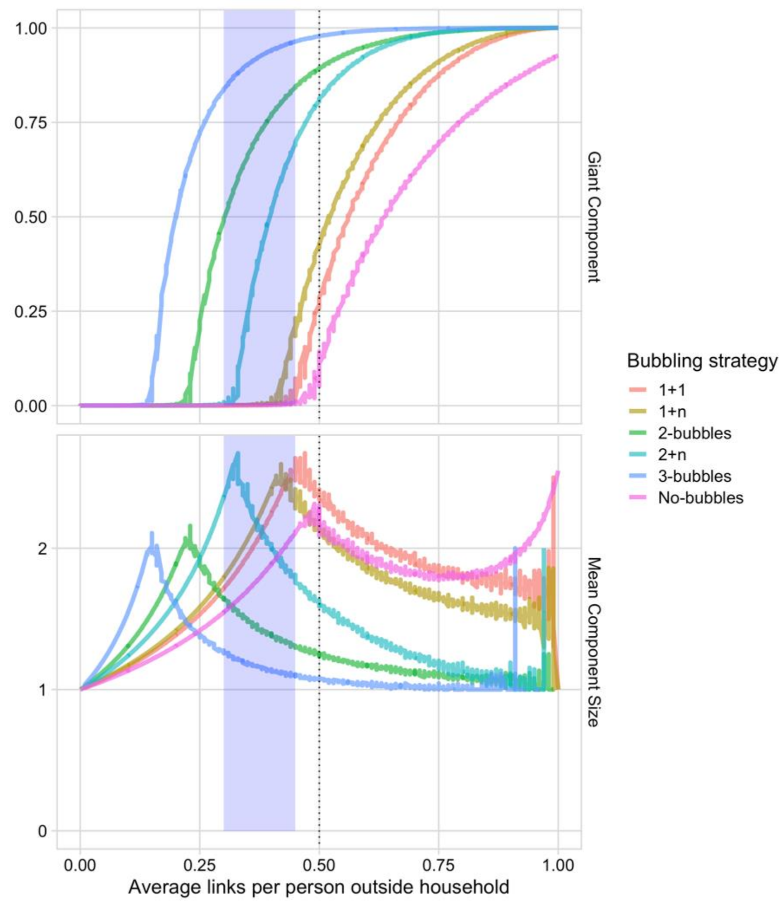

Figure 2: (Top) The proportion of households connected to the giant component for the different bubbling strategies. (Bottom) The mean component size (order parameter) for the same bubbling strategies. The shaded regions indicate an initial R_t range of between 0.6 – 0.9. A percolation threshold that lies to the right of the shaded region indicates that R_t is lower than 1 for that scenario.