Deriving the Markup

This method of deriving a markup follows exactly the framework for a Cobb-Douglas production function set out in Macallan (2008):

Notation:
$$Markup = A, output = y, labour\ input = h, fixed\ capital =$$
$$k, inputs\ of\ intermediate\ goods = x, wage = W, real\ opportunity\ cost\ of\ capital = r,$$
$$vector\ of\ prices\ for\ inputs\ of\ intermediate\ goods = q, profit\ = \pi$$

Assumption

Cobb-Douglas Production function

The fundamental assumption that is made here is that firms have a Cobb-Douglas production function. This is the benchmark production framework for a given unit of output (y), which has two or more (in this case three) inputs.

$$y = f(h, k, x) = h^\alpha k^\beta x^\gamma$$

Here:
$\alpha = elasticity\ of\ output\ with\ respect\ to\ labour, \beta =$
$elasticity\ of\ output\ with\ respect\ to\ capital, and\ \gamma =$
$elastcity\ of\ output\ with\ respect\ to\ intermediate\ goods$

Firm's Optimisation Problem

A firm's objective is to maximise its profits subject to the constraints of the production function and the demand curve it faces for its product:

i.e. Maximise $\pi = Py - Wh - rPk - qx$ **(1)**   [the firm's profit is its revenue from output less the costs of the inputs, whereby units of labour, capital and intermediate goods are multiplied by their respective unit costs]

Subject to $y = f(h, k, x)$ **(2)**   [the firm's production function- output has to be determined by the amount of labour, capital and intermediate goods]

And $y = g(P)$ **(3)**   [the firm's demand function- output is a function of price]

Solving the Optimisation Problem

Marginal Product of inputs

This problem can be solved by constructing a Lagrangian (which expresses the profit function and both constraints within one equation):

$$L = Py - Wh - rPk - qx + \lambda(f(h,k,x)) + \lambda(g(P))$$

The equation can then be differentiated with respect to labour, capital, and intermediate goods, in order to calculate the 'first order conditions', which will provide the marginal products of labour, capital and intermediate inputs respectively:

$$\frac{\partial L}{\partial h} = -W + \lambda \cdot fh(h,k,x) = 0$$

$$\frac{\partial L}{\partial k} = -rP + \lambda \cdot fk(h,k,x) = 0$$

$$\frac{\partial L}{\partial x} = -q + \lambda \cdot fx(h,k,x) = 0$$

The first derivatives of the production functions are given as $fi(h,k,x)$, where i represents the input that the production function has been differentiated by.

The lagrange multiplier (λ) can be constrained to equate to $\left(\frac{P}{A}\right)$; therefore the marginal products of the inputs and present the FOCs with respect to the firm's choice of inputs are as follows:

$$-W + \lambda \cdot fh(h,k,x) = 0 \rightarrow -W + \frac{P}{A} \cdot fh(h,k,x) = 0 \rightarrow \frac{WA}{P} = fh(h,k,x) \text{ (4)}$$

$$-rP + \lambda \cdot fk(h,k,x) = 0 \rightarrow -rP + \frac{P}{A} \cdot fk(h,k,x) = 0 \rightarrow rA = fk(h,k,x) \text{ (5)}$$

$$-q + \lambda \cdot fx(h,k,x) = 0 \rightarrow -q + \frac{P}{A} \cdot fx(h,k,x) = 0 \rightarrow \frac{qA}{P} = fx(h,k,x) \text{ (6)}$$

Marginal Costs

In addition to maximising profits, the firm has an additional constraint in that they also need to minimise costs subject to the production function. The solution to this problem implies the firm's cost function is $C(W, rP, q, y)$ - the cost incurred by a firm is a function of the unit labour costs (W), unit capital costs (rP), the unit intermediate goods costs (q) and the output produced (y).

Differentiating the cost function with respect to output obtains the marginal cost:

$$MC = \frac{\partial C(W, rP, q, y)}{\partial y} = \frac{\partial C(W, rP, q)}{\partial y}$$

The unit costs can be written as functions of the units used; thus the marginal costs can be written as:

$$MC = \frac{\partial C(W, rP, q)}{\partial y} = W\frac{\partial h}{\partial y} + rP\frac{\partial k}{\partial y} + q\frac{\partial x}{\partial y} \text{ (7)}$$

This intuitively means that the marginal cost of production equates to the sum of the marginal costs of the individual inputs. So for example, when looking at labour, $\frac{\partial h}{\partial y}$ represents the change in labour with relation to a change in output. By multiplying this by W, the marginal cost of labour is obtained.

The production function in (2) can be totally differentiated to obtain:

$$dy = fh(h,k,x)dh + fk(h,k,x)dk + fx(h,k,x)dx \text{ (8)}$$

[This follows the general form of total differentiation whereby for the function $z = f(x,y)$, the total differential $\boldsymbol{dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy}$]. This is done to allow for all variables to change with output.

Identifying the markup

The total differential can be substituted into the marginal cost equation (7):

$$MC = W\frac{\partial h}{\partial y} + rP\frac{\partial k}{\partial y} + q\frac{\partial x}{\partial y} = \frac{W\partial h + rP\partial k + q\partial x}{\partial y} = \frac{Wdh + rPdk + qdx}{fh(h,k,x)dh + fk(h,k,x)dk + fx(h,k,x)dx} \text{ (9)}$$

Further substitution, using equations (4), (5) and (6), obtains a fully expanded equation of marginal cost.

Equation (4) can be rewritten as $W = \frac{P}{A} \cdot fh(h,k,x)$

Equation (5) can be rewritten as $r = \frac{1}{A} \cdot fk(h,k,x)$

Equation (6) can be rewritten as $q = \frac{P}{A} \cdot fx(h,k,x)$

Thus marginal cost can be expanded to:

$$MC = \frac{\frac{P}{A}\cdot fh(h,k,x)dh + P\cdot\frac{1}{A}\cdot fk(h,k,x)dk + \frac{P}{A}\cdot fx(h,k,x)dx}{fh(h,k,x)dh + fk(h,k,x)dk + fx(h,k,x)dx}$$

which can then be simplified to:

$$MC = \frac{\frac{P}{A}(fh(h,k,x)dh + fk(h,k,x)dk + fx(h,k,x)dx)}{(fh(h,k,x)dh + fk(h,k,x)dk + fx(h,k,x)dx)}$$

and terms can be cancelled to yield the result:

$$MC = \frac{P}{A} \text{ (10)}$$

Intuitively, once rearranged to $P = MC \cdot A$, the output price is shown to equate to the marginal cost multiplied by a markup (A).

Introducing the concept of elasticity

The markup (A) can be rewritten in terms of the labour share (s), and the elasticity of output with respect to labour input (which from the assumption regarding the Cobb-Douglas production function is a constant, α). The labour share equates to the proportion of the <u>value</u> of output that is distributed to labour in the form of labour compensation, while α can be thought of as the responsiveness of output with respect to a change in labour (crucially, this differs to labour productivity). This can be written mathematically as $\alpha = \frac{dy}{dh} \cdot \frac{h}{y}$

Multiplying both sides of **(4)** by $\frac{hy}{hy}$ results in:

$$\frac{hy}{hy} \cdot A = \frac{hy}{hy} \cdot \frac{Pf\mathrm{h}(h,k,x)}{W} \rightarrow A = \frac{Phyf\mathrm{h}(h,k,x)}{Why} = \frac{hf\mathrm{h}(h,k,x)}{y} \cdot \frac{Py}{Wh} \quad \textbf{(11)}$$

Note that in the National Accounts framework, $\frac{Py}{Wh}$ can be thought of as total current price output (or Gross Value Added) divided by 'compensation of employees', which equates to the inverse of the labour share $\left(\frac{1}{s}\right)$. Also recall that $f\mathrm{h}(h,k,x)$ is the differential of the production function with respect to labour, which can be written as $\frac{dy}{dh}$. In addition, **(11)** can be rearranged in terms of the elasticity of output with respect to labour input, α:

$$f\mathrm{h}(h,k,x) \cdot \frac{h}{y} = \frac{dy}{dh} \cdot \frac{h}{y} = \alpha$$

Thus:

$$A = \frac{hf\mathrm{h}(h,k,x)}{y} \cdot \frac{Py}{Wh} \rightarrow A = f\mathrm{h}(h,k,x) \cdot \frac{h}{y} \cdot \frac{Py}{Wh} = \frac{\alpha}{s} \quad \textbf{(12)}$$

From the Cobb-Douglas assumption made at the start of this note, the elasticity of output with respect to labour is a constant (α). Therefore the markup (A) is inversely proportional to the labour share:

$$A = \frac{\alpha}{s}$$

As we are scaling the inverse labour share by a constant factor (α), the change in the markup will solely equate to the change in the labour share (the value of the parameter α makes no impact on the price index as it is assumed to not vary over time).

References

Macallan, C, "The cyclicality of markups and profit margins for the United Kingdom: some new evidence," *Bank of England Working Paper No. 351*, Aug. 2008.