A Class of Association Sensitive Multidimensional Welfare Indices*

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Abstract

The last few decades have seen increased theoretical and empirical interest in multidimensional measures of welfare, including various physical quality of life measures and the Human Development Index. Most multidimensional measures, though, ignore the inequality with which the various dimensions are distributed. A concern for inequality in the multidimensional setting can take two distinct forms. The first pertains to the spread of each dimensional achievements across the population, as would be reflected in the multidimensional version of the usual Lorenz criterion. The second regards correlation across dimensions, reflecting the key observation that dimensional interactions may alter evaluations of individual welfare as well as overall inequality. Recent measures have incorporated the first form of inequality, but are silent on the second form. This paper proposes a two-parameter class of multidimensional welfare indices that are sensitive to both. The properties of the new class are investigated, and it is shown that other multidimensional indices, such as the ones developed by Bourguignon (1999) and Foster, Lopez-Calva, and Szekely (2005), are sub-classes of this new broader class. The indices are applied to the year 2000 Mexican census data to explore how policy recommendations might be affected.

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1 Introduction

Measurement of social welfare has always been a challenging task for economic theorists and policy makers across the globe. It is now universally agreed that economic affluence, often measured in terms of income, can not be an exclusive indicator of social welfare, as it completely ignores the importance of other aspects, such as education, health, and housing. Since the advent of the basic needs approach and the capability approach due to Amartya Sen, the measurement of social welfare has been multidimensional in nature, which have inspired various multidimensional indices.

The early social welfare indices are either simple or weighted average of dimension specific indicators that are again obtained by the simple average of achievements across persons. This simple method of aggregation makes these indices completely insensitive to inequality across persons. As widening inequality is ethically detrimental to a society's welfare, the additive indices certainly ignore a crucial aspect of welfare measurement. Besides, an inequality sensitive social welfare index encourages any policy maker to undertake distributive policies.

There are two distinct forms of inequality in the multidimensional framework. The first form pertains to the spread of each dimensional achievements across population (Kolm 1977). In contrast, the second form is related to correlation or more precisely - association among dimensions, which is pioneered by Atkinson and Bourguignon (1982). The first form of inequality is called *distribution sensitive inequality*; whereas, the second is *association sensitive inequality*. In this paper, we develop a two-parameter class of social welfare indices that are sensitive to both forms of multidimensional inequality. We further show how these indices play a crucial role in guiding more accurate policy prescriptions.

The first form of inequality is important of welfare analysis because it encourages a policy maker to focus at the lower end of the distributions; whereas, the second is important for two distinct reasons. Primarily, if all dimensions are strongly positively associated, single-dimensional and multidimensional comparisons become more aligned. Conversely, less association among dimensions makes multidimensional analysis more informative. The second importance stems from the fact that it can advise appropriate policy recommendations when the first form fails to do so. A *distribution sensitive index* persuades a policy maker to pay attention towards the lower end of the distributions, but fails to implement proper policies when persons at the lower end of every distribution are not the same person. An association sensitive social welfare index can be helpful in this situation. However, proper knowledge about the relationship among dimensions is required as dimensions could either be substitutes or complements to each other. The relation between the concept of correlation among dimensions is related to the notion of complementarity and substitutability (Bourguignon and Chakravarty 2003).

Over the past few decades, various classes of multidimensional poverty and inequality indices have been proposed that incorporate both forms of inequality (Tsui 1995, Tsui 1999, Bourguignon 1999, Tsui 2002, Bourguignon and Chakravarty 2003, Decancq and Lugo 2008). However, only a few explicit multidimensional welfare indices have been developed incorporating either one (Hicks 1997, Foster, Lopez-Calva, and Szekely 2005) or both (Bourguignon 1999) forms. The class of welfare indices that we introduce in this paper is based on generalized means. Apart from being sensitive to both forms of inequality, the class of indices is subgroup consistent, which means that increase in welfare of one group leads to increase in the overall welfare, while that of other group unaltered. However, nothing comes without a trade-off. The class of indices that is sensitive to association in strict sense is not decomposable by dimensions. It is not possible to calculate the contribution of each dimension to total welfare. Is it a drawback? As Gajdos and Weymark (2005) points out, it depends on what one wants to accomplish with these indices. If the objective is to find only the dimensional contribution to total social welfare, one must not use strictly association sensitive social welfare indices. However, if policy recommendations are of utmost importance, then omission of association sensitive inequality is a serious drawback. Finally, the proposed class of welfare indices include various other existing classes of welfare indices as special cases.

The sections of this paper are organized as follows. In the second section, we introduce the notation. We propose the class of indices based on generalized means in the third section. The fourth section outline the non-distributional axioms and characterize the functional form of the proposed class of indices. The fifth and the sixth sections introduce the association sensitive axioms and the distributions sensitive axioms, respectively, and derive proper restrictions on parameters. The seventh section is devoted to the discussion on the class of association sensitive indices and relate the proposed class to the existing classes of indices. The eighth section discusses how sensitivity to association, in particular, influence policy recommendation. The ninth section applies the new indices to the Mexican census data for the year 2000 for policy recommendation purpose. The final section concludes this paper.

2 Notation

In this section, we introduce notations that are used throughout this paper. Let $\mathbb{R}^k$ denote the Euclidean k-space, and $\mathbb{R}^k_+, \mathbb{R}^k_{++} \subset \mathbb{R}^k$ denote the non-negative and the strictly positive k-spaces, respectively. Let $\mathbb{N}$ stand for the set of positive integers, $\mathbf{N} = \{1, \ldots, N\} \subset \mathbb{N}$ represents the set of N persons, and $\mathbf{D} = \{1, \ldots, D\} \subset \mathbb{N}$ is the set of fixed number of D dimensions. For every $M \in \mathbb{N}$ and for every $x, y \in \mathbb{R}^M$, we define $(x \vee y) = (\max(x_1, y_1), \ldots, \max(x_M, y_M))$ and $(x \wedge y) = (\min(x_1, y_1), \ldots, \min(x_M, y_M))$. For every $M \in \mathbb{N}$, any weight vector is denoted by $a \in \mathbb{R}^M_+$ such that $\Sigma_{m=1}^M a_m = 1$ and any equal weight vector is denoted by $\bar{a} \in \mathbb{R}^M_+$ such that $\bar{a}_m = 1/M \; \forall m = 1, \ldots M$. For every $M, r \in \mathbb{N}$ and for every $z \in \mathbb{R}^M$, $[z]_r$ is a replication vector where z is replicated r times such that $[z]_r = (z, \ldots, z) \in \mathbb{R}^{r \times M}$. Likewise, for every $Y \in \mathbb{R}^{LM}$, $[Y]_r$ is a replication matrix where Y is replicated r times such that $[Y]_r \in \mathbb{R}^{(r \times L)M}$. For every $L, M \in \mathbb{N}$, $\mathbf{1}_{LM} \in \mathbb{R}^{LM}$ is a matrix every element equal to 1. Similarly, for every $M \in \mathbb{N}$, $\mathbf{1}_M \in \mathbb{R}^M$ is a vector with every element equal to 1.

For any $\mathbf{N} \subset \mathbb{N}$, an achievement[1] matrix is denoted by $H \in \mathbb{R}^{ND}_{++}$ and the set of all such matrices by $\mathcal{H} = \cup_{\mathbf{N} \subset \mathbb{N}} \mathbb{R}^{ND}_{++}$. An achievement matrix with a fixed number of population $N \in \mathbb{N}$ is denoted by H_N and the set of all such matrices by $\mathcal{H}_N = \cup \mathbb{R}^{ND}_{++}$. Let h_{nd}, the nd^{th} element in H, be the achievement of person n in dimension $d \; \forall n \in \mathbf{N}$, $d \in \mathbf{D}$. Row n and column d in H are denoted by $h_{n\cdot}$ and $h_{\cdot d} \; \forall n \in \mathbf{N}$, $d \in \mathbf{D}$. A social welfare index is defined

[1] We begin with the assumption that achievements are normalized in some way or the other.

by $W : H \to \mathbb{R}$. For A society $\mathcal{A}$ has weakly (strictly) higher social welfare than another society $\mathcal{B}$ if and only if $W\left(H^{\mathcal{A}}\right) \geq (>)\, W\left(H^{\mathcal{B}}\right)$ for any $H^{\mathcal{A}}, H^{\mathcal{B}} \in \mathcal{H}$.

3 A Class of Indices

In this section, a class of social welfare indices based on generalized mean is proposed. Thus, at the beginning, we define the generalized mean[2]. For every $M \in \mathbb{N}$, for every $x \in \mathbb{R}^M_{++}$, for every $a \in \mathbb{R}^M_{+}$, and for every $\gamma \in \mathbb{R}$, the generalized mean of order γ is defined by:

$$\mu_\gamma(x;a) = \begin{cases} \left(\sum_{m=1}^M a_m x_m^\gamma\right)^{1/\gamma} & \text{for } \gamma \neq 0 \\ \prod_{m=1}^M x_m^{a_m} & \text{for } \gamma = 0 \end{cases}.$$

For $\gamma = 1$, the generalized mean reduces to weighted arithmetic mean. The generalized mean is equivalent to weighted geometric mean if $\gamma = 0$. Finally, it becomes weighted harmonic mean if $\gamma = -1$.

A special situation of generalized mean appears when all elements in a vector are weighted equally. Thus, for every $M \in \mathbb{N}$, for every $x \in \mathbb{R}^M_{++}$, for $\bar{a} \in \mathbb{R}^M_{+}$, and for every $\gamma \in \mathbb{R}$, the equal weighted version of generalized means of order γ is defined by:

$$\mu_\gamma(x;\bar{a}) = \begin{cases} \left(\frac{1}{M}\sum_{m=1}^M x_m^\gamma\right)^{1/\gamma} & \text{for } \gamma \neq 0 \\ \left(\prod_{m=1}^M x_m\right)^{1/M} & \text{for } \gamma = 0 \end{cases}.$$

The simple arithmetic mean of any $x \in \mathbb{R}^M_{++}$ is denoted by $\mu(x;\bar{a})$; whereas, the weighted arithmetic mean is denoted by $\mu(x;a)$.

We construct the social welfare index from any achievement matrix in two steps. In the first step, we calculate the standardized achievement of each person by aggregating the achievements in all D dimensions. In the second step, the social welfare index is obtained by aggregating the standardized achievements of all persons. The standardized achievements are aggregated by function $Q : \mathbb{R}^D_{++} \to \mathbb{R}_{++}$, where Q is identical across persons and is called *individual aggregation function*. Likewise, the standardized achievements are aggregated by function $\Phi : \mathbb{R}^N_{++} \to \mathbb{R}$, where Φ is called *standardized achievement aggregation function*.

During the first step aggregation, it is often a matter of interest to the policy makers to analyze how various dimensions contribute to the standardized achievement of a person. A straight forward representation of the individual aggregation function is also easily intelligible and is simple for derivation of various statistical properties. Thus, we assume the individual aggregation function to be additively separable. For every $n \in \mathbf{N}$ and for every $h_{n\cdot} \in \mathbb{R}^D_{++}$, the individual aggregation function can be expressed as:

$$Q(h_{n\cdot}) = U\left(V_1(h_{n1}) + \ldots + V_D(h_{nD})\right). \tag{1}$$

[2]The properties of generalized mean can be found in Appendix B.

Next, for every $\mathbf{N} \subset \mathbb{N}$ and for every $H \in \mathcal{H}$, the social welfare index is defined as:

$$W(H) = \Phi\left(U\left(\sum_{d=1}^{D} V_d(h_{1d})\right), \ldots, U\left(\sum_{d=1}^{D} V_d(h_{Nd})\right)\right). \tag{2}$$

In this paper, we propose the following two-parameter class of social welfare indices based on generalized means. For every $\mathbf{N} \subset \mathbb{N}$, for every $H \in \mathcal{H}$, for every $\alpha, \beta \in \mathbb{R}$, for every $a \in \mathbb{R}_{+}^{D}$, and for $\bar{a} \in \mathbb{R}_{+}^{N}$, the proposed class of social welfare indices is defined by:

$$\mathcal{W}(H; \alpha, \beta, a, \bar{a}) = \mu_{\alpha}\left(\mu_{\beta}(h_{1\cdot}; a), \ldots, \mu_{\beta}(h_{N\cdot}; a); \bar{a}\right). \tag{3}$$

Based on various desirable axioms, we show in the following section that the class of social welfare indices in (3) is the only class that can be obtained from (2).

4 Non-Distributional Axioms

In this section, we introduce non-distributional axioms that makes a class of welfare indices easily presentable and, at the same time, technically sound. At the end of this section, we show that these axioms induces the class of social welfare indices to be the one in (2).

The first axiom prevents the level of social welfare $W(\cdot)$ to change abruptly due to a change in the achievement of any person in any dimension.

Continuity (CNT)**.** For every $\mathbf{N} \subset \mathbb{N}$ and for every $H \in \mathcal{H}$, $W(H)$ is continuous on $\mathbb{R}_{++}^{ND}$.

The next two innocuous axioms make the interpretation of the welfare indices easy and attractive. According to the first of them, if a person has equal achievement in all dimensions then there is no harm to assume that the standardized achievement is also equal to any of the achievements. Moreover, if all persons have the same level of standardized achievements then we assume the social welfare index to be equal to that standardized achievement.

Normalization (NM)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $\zeta > 0$, and for every $H \in \mathcal{H}$ such that $H = \zeta \mathbf{1}_{ND}$,

$$Q(h_{n\cdot}) = \zeta \ \forall n \in \mathbf{N} \text{ and } W(H) = \zeta.$$

Secondly, we assume that the preference is *homothetic* as it is easy to work with, and linear homogeneity, a special case of homothetic preference, makes the social welfare index easily comprehensible. Thus, according to the second of these two axioms, if all achievements are changed proportionally, the social welfare also changes by the same proportion.

Linear Homogeneity (LH)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $\delta > 0$, and for every $H_N, H_N' \in \mathcal{H}_N$ such that $H_N' = \delta H_N$,

$$W(H_N') = \delta W(H_N).$$

While measuring social welfare, identity of a person should not ethically receive any significance. The next axiom ensures that we treat all persons as being anonymous and with equal importance.

Symmetry in People (SP)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $H_N, H'_N \in \mathcal{H}_N$, and for every permutation matrix[3] $P \in \mathbb{R}_+^{NN}$ such that $H'_N = PH_N$,

$$W\left(H'_N\right) = W\left(H_N\right).$$

None of the three axioms so far allows the population of a society vary. As we often perform cross-societal comparisons, we require an axiom that allows us to compare societies with different population. The axiom of *population replication invariance* guarantees that if the population of a society is replicated several times with the respective achievement vectors unaltered, then the level of social welfare remains unchanged.

Population Replication Invariance (PRI)**.** For every $r \in \mathbb{N}$ and for every $H, H' \in \mathcal{H}$ such that $H' = [H]_r$,

$$W\left(H'\right) = W\left(H\right).$$

The next axiom is the axiom of *monotonicity*. This axiom requires that if the achievement of a person in a dimension increases, while that of the rest unaltered, the social welfare should strictly increase. This axiom also requires that the standardized achievement of a person increases owing to an increase in any of the achievements.

Monotonicity (MO)**.** (i) For every $\mathbf{N} \subset \mathbb{N}$ and for every $H_N, H'_N \in \mathcal{H}_N$ such that $H'_N \geq H_N$ and $H'_N \neq H_N$,

$$W\left(H'_N\right) > W\left(H_N\right).$$

(ii) For every $n \in \mathbf{N}$, for every $h_n, h'_n \in \mathbb{R}_{++}^D$ such that $h'_n \geq h_n$ and $h'_n \neq h_n$,

$$Q\left(h'_n\right) > Q\left(h_n\right).$$

This axiom implicitly assumes that no personal achievement is harmful for the society. This axiom, however, deals with the situation when the achievement of a single person in any dimension changes. It does not take into account what happens if the welfare of an entire group of people changes. An improvement in welfare for a group of people can be accompanied by improvement in achievements of some people and deterioration in achievements for others, at the same time. The social welfare for the entire society is required to increase if the welfare of a group increases, while that of the rest unaltered.

Subgroup Consistency (SC)**.** For every $N_1, N_2, N \in \mathbb{N}$ such that $N_1 + N_2 = N$, for every $H_{N_1}, H'_{N_1} \in \mathcal{H}_{N_1}$, and for every $H_{N_2}, H'_{N_2} \in \mathcal{H}_{N_2}$, if $W\left(H'_{N_1}\right) > W\left(H_{N_1}\right)$ and $W\left(H'_{N_2}\right) = W\left(H_{N_2}\right)$, then $W\left(H'_{N_1}, H'_{N_2}\right) > W\left(H_{N_1}, H_{N_2}\right)$.

[3] A permutation matrix is a square matrix with each row and column have exactly one element equal to one and rest equal to zero. An identity matrix is a special type of permutation matrix.

Based on the set of non-distributional axioms, we characterize the class of social welfare indices in (3) by Theorem 1. We show that the functional form of the social welfare indices in (3) is both necessary and sufficient for the social welfare functions of the form in (2) to satisfy all non-distributional axioms.

Theorem 1 *For every* $\mathbf{N} \in \mathbb{N}$ *and for every* $H \in \mathcal{H}$*, a social welfare index of the form in* (2) *satisfies CNT, NM, LH, SP, PRI, MO, and SC if and only if it is of the form:*

$$\mathcal{W}(H; \alpha, \beta, a, \bar{a}) = \mu_\alpha\left(\mu_\beta(h_{1\cdot}; a), \ldots, \mu_\beta(h_{N\cdot}; a); \bar{a}\right)$$

for all $\alpha, \beta \in \mathbb{R}$*, for every* $a \in \mathbb{R}^D_+$*, and* $\bar{a} \in \mathbb{R}^N_+$*.*

Proof. See Appendix A. ■

Thus, we derived the functional form of the social welfare index that satisfies all of the non-distributional axioms introduced in this section. However, none of them discuss about inequality across persons. A policy maker who measures social welfare with an index insensitive to inequality has no incentive to undertake any distributive policy. Moreover, lesser inequality, ethically, increases the welfare of a society (Foster and Sen 1997). Therefore, it is crucial that any social welfare index is sensitive to inequality across persons. There are two distinct forms of inequality in the multidimensional context. The first form is due to Kolm (1977); whereas, the second is pioneered by Atkinson and Bourguignon (1982). The sensitivity to the first form is called *distribution sensitivity* and the sensitivity to the second is called *association sensitivity*. In the following two sections, we discuss both forms of multidimensional inequality, introduce the corresponding axioms, and show that proper restrictions on parameters α and β allow the functional form in (3) to be sensitive to both form of multidimensional inequality.

5 Association Sensitive Axioms

That inter-dimensional association is important has already been emphasized repeatedly in the previous studies (Tsui 1995, Tsui 1999, Bourguignon 1999, Tsui 2002, Bourguignon and Chakravarty 2003). An achievement matrix can be obtained from another one by increasing correlation among dimensions in various ways. Role of inter-dimensional correlation in the study of welfare analysis has first been introduced by Atkinson and Bourguignon (1982), where correlation between two dimensions can be increased, leaving the marginal distribution of dimensions unaltered. Later, a concept called *Correlation Increasing Switch* (CIS) has been defined by Bourguignon and Chakravarty (2003) with the same objective. Tsui, on the other hand, defined a concept called *Correlation Increasing Transfer* (CIT) following the notion of *basic rearrangement* due to Boland and Proschan (1988). We pursue the approach developed by Boland and Proschan[4] (1988) and call it *association*

[4] The definition of CIT and CIS are equivalent if there are only two dimensions. However, if there are more than two dimensions, the definition of CIS is bit confusing.

increasing transfer[5]. The formal definition of the concept is provided next.

For every $\mathbf{N} \subset \mathbb{N}$ and for every $H'_N, H_N \in \mathcal{H}_N$, H'_N is obtained from H_N by an association increasing transfer if $H'_N \neq H_N$, H'_N is not a permutation of H_N, and there exist two persons n_1 and n_2 such that $h'_{n_1\cdot} = (h_{n_1\cdot} \vee h_{n_2\cdot})$, $h'_{n_2\cdot} = (h_{n_1\cdot} \wedge h_{n_2\cdot})$, and $h'_{n\cdot} = h_{n\cdot}$ $\forall n \neq n_1, n_2$. Intuitively, association among dimensions increases if one person has strictly higher achievement in some dimensions but strictly lower in others before transfer, and obtains higher achievement in all dimensions than the other does after transfer[6]. Based on the concept of association increasing transfer, we develop the following set of axioms on association sensitivity of welfare indices.

Strictly Decreasing under Increasing Association (SDIA)**.** For every $\mathbf{N} \subset \mathbb{N}$ and for every $H'_N, H_N \in \mathcal{H}_N$ such that H'_N is obtained from H_N by a finite sequence of association increasing transfers,

$$W(H'_N) < W(H_N).$$

Strictly Increasing under Increasing Association (SIIA)**.** For every $N \in \mathbb{N}$ and for every $H'_N, H_N \in \mathcal{H}_N$ such that H'_N is obtained from H_N by a finite sequence of association increasing transfers,

$$W(H'_N) > W(H_N).$$

The corresponding weak versions of the axioms are *weakly decreasing under increasing association* (WDIA) and *weakly increasing under increasing association* (WIIA), which include the case $W(H') = W(H)$ along with the strict inequalities. The next obvious question is what restrictions on parameters α and β enable the proposed class on welfare indices satisfy the association sensitivity axioms. The expanded form of the proposed class of indices in (3) can be expressed as:

$$\mathcal{W}(H;\alpha,\beta,a,\bar{a}) = \begin{cases} \left(\frac{1}{N}\sum_{n=1}^{N}\left[\sum_{d=1}^{D} a_d h_{nd}^{\beta}\right]^{\alpha/\beta}\right)^{1/\alpha} & \text{if } \alpha \neq 0, \beta \neq 0 \\ \left(\frac{1}{N}\sum_{n=1}^{N}\left[\prod_{d=1}^{D} h_{nd}^{a_d}\right]^{\alpha}\right)^{1/\alpha} & \text{if } \alpha \neq 0, \beta = 0 \\ \left(\prod_{n=1}^{N}\left[\sum_{d=1}^{D} a_d h_{nd}^{\beta}\right]^{1/\beta}\right)^{1/N} & \text{if } \alpha = 0, \beta \neq 0 \\ \left(\prod_{n=1}^{N}\left[\prod_{d=1}^{D} h_{nd}^{a_d}\right]\right)^{1/N} & \text{if } \alpha = 0, \beta = 0 \end{cases}. \quad (4)$$

For our purpose of deriving the results in this section, we break down $\mathcal{W}(\cdot)$ into the following functional forms:

$$\mathcal{W}(\cdot) = \mathcal{F}(F(G(\cdot))).$$

[5] We prefer using the term 'association' rather than the term 'correlation' since term 'association' is much broader than term 'correlation'. In statistical literature, correlation simply means Pearson's product moment correlation.

[6] The definition of CIS is different in the sense that it considers increasing correlation between two dimensions only.

Based on different values of α and β, the various functional forms are summarized in Table 1.

Table 1: Different Functional Forms of $\mathcal{W}$ Break Up

	$\mathcal{F}(\cdot)$	$F(\cdot)$	$G(h_{n\cdot})$
Case I: $\alpha \neq 0,\ \beta \neq 0$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^{N} G(\cdot)$	$\mu_{\beta}^{\alpha}(h_{n\cdot};a)$
Case II: $\alpha \neq 0,\ \beta = 0$	$\left(\frac{1}{N}F(\cdot)\right)^{1/\alpha}$	$\sum_{n=1}^{N} G(\cdot)$	$\mu_{0}^{\alpha}(h_{n\cdot};a)$
Case III: $\alpha = 0,\ \beta \neq 0$	$(F(\cdot))^{1/N}$	$\prod_{n=1}^{N} G(\cdot)$	$\mu_{\beta}(h_{n\cdot};a)$
Case IV: $\alpha = 0,\ \beta = 0$	$(F(\cdot))^{1/N}$	$\prod_{n=1}^{N} G(\cdot)$	$\mu_{0}(h_{n\cdot};a)$.

Based on Table 1, we need to obtain the restrictions on parameters that enable $\mathcal{W}(\cdot)$ to be association sensitive. To establish the desired results, we resort to the lattice theory. The following definition introduces strict L-subadditivity, L-superadditivity, and valuation[7].

Definition 1 *For every $M \in \mathbb{N}$, for every $x, y \in \mathbb{R}_{+}^{M}$, (i) any function $\mathbf{G}$ is strict L-subadditive if $\mathbf{G}(x \vee y) + \mathbf{G}(x \wedge y) < \mathbf{G}(x) + \mathbf{G}(y)$, (ii) any function $\mathbf{G}$ is strict L-superadditive if $\mathbf{G}(x \vee y) + \mathbf{G}(x \wedge y) > \mathbf{G}(x) + \mathbf{G}(y)$, and (iii) any function $\mathbf{G}$ is a valuation if $\mathbf{G}(x \vee y) + \mathbf{G}(x \wedge y) = \mathbf{G}(x) + \mathbf{G}(y)$.*

First, we show how $F(\cdot)$ is behaves due to association increasing transfers, which depends on whether $G(\cdot)$ is L-subadditive, L-superadditive, or valuation. It is apparent from Table 1 that there are two distinct functional forms for $F(\cdot)$. The first is additive, i.e., $F(\cdot) = \sum_{i=1}^{n} G(\cdot)$, corresponding to $\beta \neq 0$ and the second is multiplicative, i.e., $F(\cdot) = \prod_{i=1}^{n} G(\cdot)$, corresponding to $\beta = 0$. The following proposition is due to Boland and Proschan (1988), Proposition 2.5 (b).

Proposition 1 *For every $\mathbf{N} \subset \mathbb{N}$, for every $\mathbf{F}(H) = \sum_{n=1}^{N} \mathbf{G}(h_{n\cdot})$, and for every $H_N, H_N' \in \mathcal{H}_N$ such that H_N' is obtained from H_N by a finite sequence of association increasing transfers, (i) $\mathbf{F}(H_N) < \mathbf{F}(H_N')$ if and only if $\mathbf{G}(\cdot)$ is strict L-subadditive on $\mathbb{R}_{+}^{D}$, (ii) $\mathbf{F}(H_N) > \mathbf{F}(H_N')$ if and only if $\mathbf{G}(\cdot)$ is strict L-superadditive on $\mathbb{R}_{+}^{D}$, and (iii) $\mathbf{F}(H_N) = \mathbf{F}(H_N')$ if and only if $\mathbf{G}(\cdot)$ is a valuation on $\mathbb{R}_{+}^{D}$.*

Proof. See Boland and Proschan (1988). ■

Proposition 1 deals with the situation where $F(\cdot)$ has the additive form. The following corollary considers the situation when $F(\cdot)$ has the multiplicative form.

[7] L-subadditive and L-superadditive stands for Lattice subadditive and Lattice superadditive, respectively; whereas, valuation implies both Lattice subadditive and Lattice superadditive.

Corollary 1 *For every $\mathbf{N} \subset \mathbb{N}$, for every $\mathbf{F}(H_N) = \prod_{n=1}^{N} \mathbf{G}(h_{n\cdot})$, and for every $H_N, H'_N \in \mathcal{H}_N$ such that H'_N is obtained from H_N by a finite sequence of association increasing transfers, (i) $\mathbf{F}(H_N) < \mathbf{F}(H'_N)$ if and only if $\log \mathbf{G}(\cdot)$ is strict L-subadditive on $\mathbb{R}^D_+$, (ii) $\mathbf{F}(H_N) > \mathbf{F}(H'_N)$ if and only if $\log \mathbf{G}(\cdot)$ is strict L-superadditive on $\mathbb{R}^D_+$, and (iii) $\mathbf{F}(H_N) = \mathbf{F}(H'_N)$ if and only if $\log \mathbf{G}(\cdot)$ is a valuation on $\mathbb{R}^D_+$.*

Proof. Let H_N,$H'_N \in \mathcal{H}_N$ for any $\mathbf{N} \subset \mathbb{N}$. There exist persons $n_1, n_2 \in \mathbf{N}$ and $n_1 \neq n_2$, such that $h_{n_1\cdot} \ngtrless h_{n_2\cdot}$. H'_N is obtained from H_N by a sequence of association increasing transfers. From Proposition 1, we know that $\log \mathbf{F}(H'_N) < \log \mathbf{F}(H_N)$ if and only if $\log \mathbf{G}(\cdot)$ is strict L-subadditive. As $\log \mathbf{F}(\cdot)$ is a monotonic transformation of $\mathbf{F}(\cdot)$, it follows that $\mathbf{F}(H'_N) < \mathbf{F}(H_N)$. The other two parts can be proved in identical fashion. ■

From Proposition 1 and Corollary 1, it follows how sensitive $F(\cdot)$ is to association increasing transfers under different circumstances. The remaining task is to implement the proposition and the corollary on the proposed class of indices to derive the restrictions on α and β. To do that, we need to figure out beforehand the restrictions on α and β that enables $G(\cdot)$ to be strictly L-subadditive, strictly L-superadditive, and valuation. The following definition is helpful for this purpose.

Definition 2 *For any twice differentiable function $\mathbf{G} : R^D_{++} \to R_+$, (i) strict L-subadditivity requires all cross partial derivatives to be negative, i.e. $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} < 0 \ \forall d_1 \neq d_2$; (ii) strict L-superadditivity requires $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} > 0 \ \forall d_1 \neq d_2$; (iii) valuation requires $\partial^2 \mathbf{G}(h_{n\cdot}) / \partial h_{nd_1} \partial h_{nd_2} = 0 \ \forall d_1 \neq d_2$.* (*Topkis* 1998)

All functional forms of $G(h_{n\cdot})$ in Table 1 are various forms of generalized mean and are twice differentiable for $h_{nd} \in \mathbb{R}_{++} \ \forall n, d$. Table 2 summarizes the restrictions on α and β that allow $G(\cdot)$ and $\log G(\cdot)$ to satisfy strict L-subadditivity, strict L-superadditivity, and valuation (Please see the supplemental file for the functional forms of the second order differentiations).

Table 2: Restrictions on α and β

Strict L-subadditive	Strict L-superadditive	Valuation
$\alpha < 0$ & $\alpha > \beta$	$\alpha < 0$ & $\alpha < \beta$	$\alpha = \beta$
$\alpha > 0$ & $\alpha < \beta$	$\alpha > 0$ & $\alpha > \beta$	
$\alpha = 0$ & $\beta > 0$	$\alpha > 0$ & $\beta = 0$	
	$\alpha < 0$ & $\beta = 0$	
	$\alpha = 0$ & $\beta < 0$	

Based on the restrictions on the parameters in Table 2 and based on Proposition 1 and Corollary 1, we obtain the following Theorem.

Theorem 2 *For every* $\mathbf{N} \subset \mathbb{N}$*, for every* $H \in \mathcal{H}$*, for every* $a \in \mathbb{R}_+^D$*, for* $\bar{a} \in \mathbb{R}_+^N$*, and for every* $\alpha, \beta \in \mathbb{R}$*, (i)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies SDIA if and only if* $\alpha < \beta$*. (ii)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies SIIA if and only if* $\alpha > \beta$*.*

Proof. See Appendix A. ■

Theorem 2 imposes further restrictions on parameters α and β that enable the proposed class of welfare indices to be association sensitive in strict sense[8]. In this section, we obtained the restrictions on parameters that would allow the proposed class of indices to be sensitive to one form of multidimensional inequality. The next section is devoted to the sensitivity to the other form of multidimensional inequality.

6 Distribution Sensitive Axioms

The other form of multidimensional inequality is distribution sensitive inequality. This is based on the concept of distribution sensitivity in the single dimensional context where $D = 1$. A unidimensional distribution is stated to be more equal than another distribution, if the former is obtained from the latter by a finite sequence of Pigou-Dalton transfers. Distribution $x = (x_1, \ldots, x_N)$ is obtained from another distribution $y = (y_1, \ldots, y_N)$ by a Pigou-Dalton transfer if $x \neq y$, $x_{n_1} = \lambda y_{n_1} + (1 - \lambda) y_{n_2}$, $x_{n_2} = (1 - \lambda) y_{n_1} + \lambda y_{n_2}$, and $x_n = y_n \; \forall n \neq n_1, n_2$, where $\lambda \in (0, 1)$. Alternatively, a distribution x is obtained from another distribution y by a finite sequence of Pigou-Dalton transfers if and only if $x = By$, where B is a bistochastic matrix[9]. Therefore, if a distribution is obtained from another distribution by multiplying by a bistochastic matrix, the former is more equal. The concept of *common smoothing* builds on the multidimensional extension of this construct. For every $\mathbf{N} \subset \mathbb{N}$ and for every $H'_N, H_N \in \mathcal{H}_N$, H'_N is obtained from H_N by common smoothing if there exists a bistochastic matrix B, which is not a permutation matrix, such that $H'_N = BH_N$. Note that a finite sequence of Pigou-Dalton transfers in the multidimensional context is not equivalent to common smoothing. This equivalence relation holds for $N = 2$ or $D = 1$, but does not hold both ways for $N \geq 3$ and $D \geq 2$. The following two axioms relate the notion of common smoothing to distribution sensitive multidimensional social welfare indices.

Strictly Increasing under Common smoothing (SICS)**.** For every $\mathbf{N} \subset \mathbb{N}$, for every $H'_N, H_N \in \mathcal{H}_N$, and for every bistochastic matrix $B \in \mathbb{R}_+^{NN}$ such that $H'_N = BH_N$,

$$W(H'_N) > W(H_N).$$

The weak version of this axiom is *weakly increasing under common smoothing* (WICS) that includes the case $W(H'_N) = W(H_N)$ apart from the strict inequality. Now, we need

[8] The corresponding results containing WDIA and WIIA can be easity obtained. It can be shown that $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ satisfies WDIA (WIIA) if and only if $\alpha \leq \beta$ ($\alpha \geq \beta$).

[9] A bistochastic matrix is a non-negative square matrix whose row sum and column sum are both equal to one.

to derive the set of restrictions that allows $\mathcal{W}(\cdot)$ to satisfy the axioms of distribution sensitivity. The following proposition provide the basis for deriving proper restrictions on the parameters.

Proposition 2 *For every* $\mathbf{N} \subset \mathbb{N}$, *for every* $W(H) = \Phi(Q(h_{1\cdot}), \ldots, Q(h_{N\cdot}))$, *for every* $H'_N, H_N \in \mathcal{H}_N$, *and for every bistochastic matrix* B *such that* $H'_N = BH_N$, *if* $\Phi(\cdot)$ *is non-decreasing and quasi-concave and* $Q(\cdot)$ *is concave then* $W(H'_N) \geq W(H_N)$.

Proof. See Theorem 4 and Theorem 5 of Kolm (1977). ■

Based partly on Proposition 2, the following theorem is obtained. The following theorem shows that the proposed class of indices is distribution sensitive under proper restrictions on the parameters.

Theorem 3 *For every* $N \in \mathbb{N}$, *for every* $H_N \in \mathcal{H}_N$, *for every* $a \in \mathbb{R}^D_+$, *and for* $\bar{a} \in \mathbb{R}^N_+$, *(i)* $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a})$ *satisfies SICS if and only if* $\alpha, \beta \leq 1$ *and* $\alpha = \beta \neq 1$, *(ii)* $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a})$ *satisfies WICS if and only if* $\alpha, \beta \leq 1$.

Proof. See Appendix A. ■

In the following section, we present the class of association sensitive indices.

7 The Class of Association Sensitive Welfare Indices

In the previous sections, we have imposed restrictions on α and β that enables $\mathcal{W}(H)$ to satisfy all the desirable axioms introduced until now. According to Theorem 1, it satisfies CNT, NM, LH, SP, PRI, MO, and SC for all $(\alpha, \beta) \in \mathbb{R}^2$. According to Theorem 2, $\mathcal{W}(H)$ satisfies SDIA (SIIA) if and only if $\alpha < \beta$ ($\alpha > \beta$). Further restrictions are imposed on both parameters when the class of indices is required to satisfy SICS. According to Theorem 3, $\mathcal{W}(H)$ satisfies SICS if and only if $\alpha, \beta \leq 1$ and $\alpha = \beta \neq 1$.

Based on these theorems above, the following theorem summarizes all restrictions on α and β.

Theorem 4 *For every* $\mathbf{N} \subset \mathbb{N}$, *for every* $H \in \mathcal{H}$, *for every* $a \in \mathbb{R}^D_+$, *and for* $\bar{a} \in \mathbb{R}^N_+$, *(i)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and SDIA if and only if* $\alpha < \beta \leq 1$. *(ii)* $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ *satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and SIIA if and only if* $\beta < \alpha \leq 1$.

Proof. The proof is straight forward and directly follows from Theorem 1, Theorem 2, and Theorem 3. ■

Theorem 4 shows that the proposed class of indices is strictly sensitive to both forms of inequality if and only if $\alpha, \beta \leq 1$ and $\alpha \neq \beta$. However, a careful analysis would reveal that the proposed class of indices includes various other existing class of welfare indices.

If we set $\alpha = \beta = 1$, we have the following welfare index:

$$\mathcal{W}(H; 1, 1, a, \bar{a}) = \mu(\mu(h_{1\cdot}; a), \ldots, \mu(h_{N\cdot}; a), \bar{a}). \quad (5)$$

In (5), the social welfare is a simple mean of simple weighted means and $\alpha = \beta = 1$ implies that the index does not satisfy SICS and SDIA or SIIA. As a result, the index is not sensitive to any form of inequality across persons. A further manipulation would lead to the following form:

$$\mathcal{W}(H; 1, 1, a, \bar{a}) = \mu(\mu(h_{\cdot 1}; \bar{a}), \ldots, \mu(h_{\cdot D}; \bar{a}), a). \quad (6)$$

Form (6) is very familiar as this is applied while calculating well-known welfare indices such as the human development index and the physical quality of life index ((Morris 1979)).

Next, if we set $\beta = \alpha < 1$, then we obtain the following class of indices:

$$\mathcal{W}(H; \alpha, \alpha, a, \bar{a}) = \mu_\alpha(\mu_\alpha(h_{1\cdot}; a), .., \mu_\alpha(h_{N\cdot}; a), \bar{a}). \quad (7)$$

The class in (7) are strictly distribution sensitive but are not strictly association sensitive. Again, a quick manipulation of (7) would give us another familiar class of indices:

$$\mathcal{W}(H; \alpha, \alpha, a, \bar{a}) = \mu_\alpha(\mu_\alpha(h_{\cdot 1}; \bar{a}), .., \mu_\alpha(h_{\cdot D}; \bar{a}), a). \quad (8)$$

The class of welfare indices in (8) are proposed by Foster, Lopez-Calva, and Székely (2005).

Notice that the only difference between both functional forms of $\mathcal{W}(H; 1, 1, a, \bar{a})$ and $\mathcal{W}(H; \alpha, \alpha, a, \bar{a})$ is that the sequence of aggregation has been altered. Instead of first aggregating across dimensions, if the aggregation takes place first across people, the resulting values of the indices do not change. This axiom is called *path independence* by Foster et. al. (2005).

Path Independence (PI)**.** For every $H \in \mathcal{H}$,

$$\Phi(Q(h_{1\cdot}), \ldots, Q(h_{N\cdot})) = Q(\Phi(h_{\cdot 1}), \ldots, \Phi(h_{\cdot D})).$$

According to this axiom, the sequence of aggregation does not matter. The axiom is especially important under the circumstances when the data for different dimensions are available at various disaggregated levels; for example, education data are available at the individual level, income data are available at the household level, health data are available at the municipality level etc. However, we can see from the following theorem that proposed class of indices can not satisfy axiom PI and be strictly sensitive to association at the same time.

Theorem 5 *For every $H \in \mathcal{H}$, for every $a \in \mathbb{R}^D_+$, and for $\bar{a} \in \mathbb{R}^N_+$, (i) $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ satisfies CNT, NM, LH, SP, PRI, MO, SC, and PI if and only if $\alpha = \beta$, (ii) $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ satisfies CNT, NM, LH, SP, PRI, MO, SC, SICS, and PI if and only if $\beta = \alpha < 1$.*

Proof. By Theorem 1, we know that $\mathcal{W}(H; \alpha, \beta, a, \bar{a})$ satisfies all non-distributional axioms. Define $W_1 = \mu_\alpha\left(\mu_\beta(h_{1\cdot}; a), \ldots, \mu_\beta(h_{N\cdot}; a); \bar{a}\right)$ and $W_2 = \mu_\beta(\mu_\alpha(h_{\cdot 1}; \bar{a}), .., \mu_\alpha(h_{\cdot D}; \bar{a}), a)$. It is straight forward to show that if $\alpha = \beta$ then $W_1 = W_2$. Next, we show that $W_1 \neq W_2$ if $\alpha \neq \beta$. According to Theorem 26 of Hardy, Littlewood, and Pólya (1934), $W_1 > W_2$ for $\beta < \alpha$, and $W_1 < W_2$ for $\beta > \alpha$.[10]

The second part of the theorem directly follows by further applying Theorem 3. ■

The proposed class of indices contains another existing class of indices that is suggested by Bourguignon (1999), while commenting on the class of inequality indices proposed by Maasoumi (1999). For every $H \in \mathcal{H}$, for every $a \in \mathbb{R}_+^D$, for $\bar{a} \in \mathbb{R}_+^N$, for every $0 < \alpha < 1$, and for every $\beta < 1$,

$$W_B\left(H; \alpha, \beta, a, \bar{a}\right) = \left(\mu_\alpha\left(\mu_\beta\left(h_{1\cdot}; a\right), \ldots, \mu_\beta\left(h_{N\cdot}; a\right); \bar{a}\right)\right)^\alpha = \left(\mathcal{W}\left(H; \alpha, \beta, a, \bar{a}\right)\right)^\alpha.$$

Therefore, for $\beta < 1$ and $0 < \alpha < 1$, $\mathcal{W}\left(H; \alpha, \beta, a, \bar{a}\right)$ is simply a monotonic transformation of the Bourguignon class of indices. The Bourguignon class of indices satisfies all non-distributional axioms. This class of indices is sensitive to both forms of multidimensional inequality in strict sense whenever $\alpha \neq \beta$. However, the role of the inequality aversion parameter, α (as it is called in the article), is not transparent for two reasons. First of all, suppose that there are two societies with identical achievement vectors, with perfect equality across persons, and with identical values of parameter β. The only difference is societies' aversions towards inequality. Bourguignon Index would yield different levels of social welfare for both societies. It is not transparent what causes this difference though since inequality aversion should not play any role as there exist no inequality! In addition, it is not apparent what value of the inequality aversion parameter leads to higher degree of inequality aversion. The degree of inequality aversion does not monotonically depend on the inequality aversion parameter.

8 Policy Prescription

Once we have derived the class of indices, it would be interesting to see how different restrictions on parameters influence the policy prescription for a policy maker. Suppose that the policy maker has a budget of one indivisible dollar. The policy maker's concern is *who the first person should be to receive the dollar to maximize the improvement in social welfare.* We devote this section to find the answer to this question and show how sensitivity to association among dimensions affect policy maker's decision. Following Theorem 4 and Theorem 5, for every $N \in \mathbb{N}$, for every $H \in \mathcal{H}$, for every weight vector $a \in \mathbb{R}_+^D$ and $\bar{a} = \mathbf{1}_N/N$, and for every $\alpha, \beta \leq 1$, the class of social welfare index we consider in this section is:

$$\mathcal{W}\left(H; a, \alpha, \beta\right) = \mu_\alpha\left(\mu_\beta\left(h_{1\cdot}; a\right), \ldots, \mu_\beta\left(h_{N\cdot}; a\right)\right).$$

The social welfare function is based on generalized mean and is thus is differentiable.

[10] Although Hardy, Littlewood, and Pólya proves the theorem for $\alpha, \beta > 0$, it can easily extended for all $\alpha, \beta \in \mathbb{R}$.

If the policy maker provides the assistance (\$1) to person n to improve her achievement in dimensions d, the increment in total welfare is:

$$\frac{\partial \mathcal{W}(H; a, \alpha, \beta)}{\partial \$1} = \left(a_d h_{nd}^{\beta-1} C_n^{\alpha-\beta} \mathbf{C} \right) c_{nd};$$

where a_d is the share of dimension d in social welfare, $c_{nd} = \partial h_{nd} / \partial \$1$ is the improvement in achievement h_{nd} due to the dollar, $C_n = \mu_\beta (h_{n\cdot}; a)$ is the standardized achievement, and $\mathbf{C} = (\mathcal{W}(H; a, \alpha, \beta))^{1-\alpha}$. The policy maker would assist person n to improve her achievement in dimension d if

$$\left(a_d h_{nd}^{\beta-1} C_n^{\alpha-\beta} \mathbf{C} \right) c_{nd} > \left(a_{\bar{d}} h_{\bar{n}\bar{d}}^{\beta-1} C_{\bar{n}}^{\alpha-\beta} \mathbf{C} \right) c_{\bar{n}\bar{d}} \; \forall \bar{n} \in \{1, \ldots, N\} / \{n\}, \bar{d} \in \{1, \ldots, D\} / \{d\}.$$

Define $\omega_{nd} = a_d h_{nd}^{\beta-1} C_n^{\alpha-\beta} c_{nd}$. As $\mathbf{C}$ is identical across all persons, the policy maker just needs to verify whether

$$\omega_{nd} > \omega_{\bar{n}\bar{d}} \; \forall \bar{n} \in \{1, \ldots, N\} / \{n\}, \bar{d} \in \{1, \ldots, D\} / \{d\}. \quad (9)$$

To have intuition on how policy prescriptions are affected, we make two simplifying assumptions for a while. Assume that $a_d = a_{\bar{d}} \; \forall d \neq \bar{d}$, i.e. the share of all dimensions in social welfare is the same; and $s_{nd} = s_{\bar{n}\bar{d}} \; \forall d \neq \bar{d}, n \neq \bar{n}$, i.e. improvement in all dimensions for all persons from the dollar is also the same.

First, suppose the policy maker has already decided to assist person n. As there are more than one dimension, the question is which dimension of that person she should focus on. The policy maker would spends the dollar on dimension d of person n if $\omega_{nd} = \max(\omega_{n\cdot})$. A further simplification requires that $h_{nd} = \min(h_{n\cdot})$, i.e. she provides the assistance to the dimension in which person n's achievement is the lowest.

Secondly, suppose there are two persons n and $\bar{n}$ such that $\min(h_{n\cdot}) = \min(h_{\bar{n}\cdot})$ but $C_n \neq C_{\bar{n}}$. In this situation, policy maker's decision is based on the relation among dimensions. If dimensions are substitutes, then higher correlation is bad and $\alpha < \beta$. We already know that any person, if assisted, would receive the dollar to improve the dimension where her achievement is the lowest. Therefore, person n would receive the assistance instead of person $\bar{n}$ if $C_n^{\alpha-\beta} > C_{\bar{n}}^{\alpha-\beta}$, or, $C_n < C_{\bar{n}}$. On the other hand, complementarity requires $\alpha > \beta$ and person n would receive the assistance instead of person $\bar{n}$ if $C_n > C_n$.

In the third situation, suppose both $C_n \neq C_{\bar{n}}$ and $\min(h_{n\cdot}) \neq \min(h_{\bar{n}\cdot})$. Person n would receive the dollar if $(\min(h_{n\cdot}))^{\beta-1} C_n^{\alpha-\beta} > (\min(h_{\bar{n}\cdot}))^{\beta-1} C_{\bar{n}}^{\alpha-\beta}$. The decision does not only depend on the minimum achievement but also on the overall achievement. However, if the welfare index satisfies axiom PI then $\alpha = \beta$ and person n would receive the assistance if $\min(h_{n\cdot}) < \min(h_{\bar{n}\cdot})$ only. The policy prescription remains unaffected even if C_n is reasonably larger than $C_{\bar{n}}$.

In case, the dimensions are perfect substitutes in Hicks Value and Capital sense, which requires $\beta = 1$, it can be easily shown that person n would receive the assistance instead of person $\bar{n}$ if $C_n < C_{\bar{n}}$ and it does not matter in which dimension person n has least achievement. If we move to the other extreme and assume that dimensions are perfect complements, then the assistance would be provided to dimension with least achievement. Person n will receive the dollar to improve dimension d if $h_{nd} = \min(\min(h_{1\cdot}), \ldots, \min(h_{N\cdot}))$.

Without the simplifying assumptions, the general condition for assisting person n with the dollar to improve dimension d is given by (9). In the next section, an empirical illustration is provide to show the application of the class of indices.

9 Empirical Illustration

In this section, we provide an empirical illustration to analyze the efficacy of the new class of indices in terms of policy recommendation. In the previous sections, all results were obtained at the individual level. It was implicitly assumed that the target for the policy maker was persons. However, in practice, a policy maker often needs to set targets at lesser disaggregated levels, e.g., a central government needs to focus at the state level, a local government needs to focus at the community level, and so on. There could be various reasons behind such targets. There are several dimensions such as community infrastructure, social cohesion, child mortality rate, which can not be conveniently measured at the individual level. Consequently, a welfare index based on such dimensions needs to be constructed from information at lesser disaggregated levels. In this section, we consider a situation similar to this where a state government is required to target at the municipality level. Thus, our unit of analysis is municipalities instead of persons.

We use the sample data on Mexican population census for the year 2000 and focus on state Aguascalientes[11] which has eleven municipalities. The sample data set contains information on education and income for 93,761 individuals. The infant mortality data are available only at the municipality level. The household level income are not comparable to that of the standard income measure used while constructing national welfare indices, so we apply a two step approach. In the first step, we estimate the per capita income for each household from the sample and calculate the average income for every municipality. In the second step, we raise them by factors equal to the ratio of the State level GDP per capita for the year 2000 (provided by the National Statistical Institute, Mexico), to the state level Census average income. The average income for each municipality is then normalized with respect to a minimum of 100 pesos[12] and a maximum of 226,628 pesos[13].

The education variable is constructed by combining the literacy rate and enrolment rate of the municipalities. The literacy rate is estimated by the proportion of literate persons in the age group of 14 years and older and the enrolment rate is estimated by the proportion of children and youths attending school in the age group of 6-24. The education index is estimated by an weighted average of the literacy rate and the enrolment rate. We attach 2/3 weight to the literacy rate and 1/3 weight to the enrolment rate[14]. The literacy rate and the enrolment rate are normalized between zero and one by construction. Finally, the

[11] This state has been chosen just for the purpose of this example. Any other state can be randomely chose to perform similar exercises.

[12] Following the methodology of the UNDP, we take the logarithm of income and, accordingly, restrict the lower bound of income to a positive value.

[13] This value is equivalent to USD 40,000 that is applied by the UNDP as an upper limit of per-capita-GDP. We use a deflator from the 2002 human development report.

[14] For simplicity, we just followed the standard methodology pursued to construct the Human Development Index (HDR 2006, UNDP).

municipality level infant mortality rate is used as a proxy variable for the health status of the respective municipalities. The child mortality rate is also normalized between zero and 1 with a best and worst possible values of zero and 100 children not surviving per 1,000 births[15]. The following table summarizes the municipality level normalized achievement matrix of state Aguascalientes.

Table 3: Municipality Level Normalized Achievement Matrix for State Aguascalientes

Municipality (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**Min.**
Education (E)	0.86	0.81	0.80	0.84	0.81	0.83	0.82	0.86	0.80	**0.79**	0.81	0.79
Income (I)	0.86	0.71	0.73	0.75	0.79	0.80	0.75	**0.69**	0.73	0.71	0.75	0.69
Health (H)	0.79	0.73	0.76	0.75	0.77	0.78	0.77	0.75	0.74	**0.73**	0.77	0.73

It is evident from the table that the minimum achievements in three dimensions are 0.79 (education of municipality 10), 0.69 (income of municipality 8), and 0.73 (health of municipality 10), respectively. The policy maker of state Aguascalientes needs to decide which municipality should first receive the indivisible assistance. We make the two simplifying assumptions as we did in the previous section. First, all three dimensions are weighted equally, i.e., $a_E = a_I = a_H = 1/3$; second, $c_{nE} = c_{nI} = c_{nH}$ for all $n = 1, \ldots, 11$. To decide who received the dollar, we need to calculate $\omega_{nd} = h_{nd}^{\alpha-1} C_n^{\alpha-\beta}$ for all n and $d = I, E, H$. Assume that $\alpha = -2$ and $\beta = 0.5$. This a situation, where the dimensions are assumed to be substitutes to each other and the policy maker highly inequality averse. In Table 4, we report the values of ω_{nd}. We find that the tenth municipality has the largest value in the income dimension. Therefore, the policy maker must assist the 10$^{\text{th}}$ municipality to improve the employment status of her people so that overall income is increased.

Table 4: Policy Recommendation Matrix for State Aguascalientes When $\alpha = -2$ and $\beta = 0.5$

Municipality (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**Max**
Mun-Ranks	**1**	**10**	**8**	**5**	**3**	**2**	**4**	**7**	**9**	**11**	**6**	**-**
S_n	0.84	0.75	0.76	0.78	0.79	0.80	0.78	0.76	0.75	0.74	0.78	-
ω_{nE}	1.68	2.29	2.22	2.04	1.99	1.89	2.05	2.11	2.26	2.39	2.08	2.39
ω_{nI}	1.67	2.44	2.33	2.17	2.01	1.92	2.15	2.35	2.37	**2.53**	2.16	2.53
ω_{nH}	1.75	2.40	2.28	2.17	2.04	1.95	2.13	2.27	2.35	2.49	2.13	2.49

Next, suppose social welfare is measured by a path independent welfare index that does not take into account the standardized achievement of municipalities and focus only on the dimensional achievements. Then the policy maker would always be encouraged to assist the 8$^{\text{th}}$ municipality to improve the employment status of her people, which will result in higher income. Finally, consider a situation where the policy maker treats all dimensions to be perfect substitutes but is highly averse towards inequality across persons, i.e. $\alpha = -2$ and $\beta = 1$. The policy maker should focus on the person with lowest standardized achievement in this situation. We generate the values of ω_{nd} for all n, d in Table 5 in support of our claim. The eleventh person should receive the assistance in this situation on any dimension.

[15] Child mortality is measured as the number of child not surviving per 1,000 births.

Table 5: Policy Recommendation Matrix for State Aguascalientes When $\alpha = -2$ and $\beta = 1$

Municipality (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**Max**
Mun-Ranks	**1**	**10**	**8**	**5**	**3**	**2**	**4**	**7**	**9**	**11**	**6**	**-**
S_n	0.84	0.75	0.76	0.78	0.79	0.80	0.78	0.77	0.76	0.74	0.78	-
ω_{nE}	1.70	2.37	2.27	2.12	2.01	1.92	2.11	2.22	2.32	**2.47**	2.12	2.47
ω_{nI}	1.70	2.37	2.27	2.12	2.01	1.92	2.11	2.22	2.32	**2.47**	2.12	2.47
ω_{nH}	1.70	2.37	2.27	2.12	2.01	1.92	2.11	2.22	2.32	**2.47**	2.12	2.47

We provided few examples in an extremely simplifying environment to show how policy prescriptions are affected by the use of a welfare index that is sensitive to multidimensional association apart from being sensitive to distribution. In practice, however, both of our simplifying assumptions appear to be too fancy, but does not change the intuition. Even when $a_d \neq a_{d'}$ $\forall d \neq d'$, and $s_{nd} \neq s_{n'd'}$ $\forall n, d$, a policy maker is required to create the ω_{nd} values for all n and d and provide the indivisible assistance to improve the dimension of a person that leads to largest improvement in social welfare.

10 Conclusion

In this paper, we propose a class of welfare indices that is sensitive to multidimensional association besides being sensitive to multidimensional distribution. We discuss that the policy implication and targeting becomes more competent if an welfare index is association sensitive in strict sense. Association sensitive welfare indices encourage policy makers to put emphasis on standardized achievements also, instead of merely considering the individual achievements. The new class of welfare indices are based on generalized means and, thus, easily intelligible. It is shown that the path independent welfare indices are excluded from the set of strictly association sensitive indices. We show that the proposed class of indices include several other indices, such as the class of indices proposed by Foster et. al. (2005) and Bourguignon (1999). Finally, we apply the new class of indices on the Mexican census data for the year 2000 .

Although, the new class of indices solved few existing problems, many others remain. By construction, the new class of indices still assume the same degree of elasticity of substitution among all dimensions. It is assumed that all dimensions are either substitutes or complements to each other. It is not possible for the proposed class of indices to treat few dimensions to be substitutes while the rest to be complements. Moreover, this paper only considers Boland and Proschan (1988) type association increasing transfer. Therefore, further research is required in this area to find a class of welfare measure that is robust to these problems.

Finally, in this paper, we assume that the dimensional weight vector and the appropriate value of the parameters are determined normatively by the policy makers. We also do not provide any suggestion as how to calculate the increase in dimensional achievement due to any assistance since it is out of the scope of this paper and we leave this for future research.

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Appendix A

Proof of Theorem 1. The sufficiency part of the proof is straight forward. It can be easily shown that if the social welfare function is of the form (3), then it satisfies CNT, NM, LH, SP, PRI, MO, and SC. These results follow mostly from the fact that generalized means also satisfy the set of axioms.

Next, we show that the set of axioms enable us to obtain the particular form of the social welfare index in (3), i.e., we prove the necessary part. Consider two distributions of standardize achievements, s and t, with the same population size $N > 2$,. Let us split both distributions into two subgroups of size N_1 and size $N_2 = N - N_1$ such that $s = (s_1, s_2)$ and $t = (t_1, t_2)$. It can be shown that subgroup consistency implies:

$$\Phi_N(s_1, s_2) \geq \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1, t_2) \geq \Phi_N(t_1, t_2). \quad (10)$$

Note that axiom MO and axiom CNT ensures $\Phi_N(s)$ to have only one value for any $s \in \mathbb{R}^N_{++}$. If $\Phi_N(s_1, s_2) > \Phi_N(t_1, s_2)$, then by SC, we can never have $\Phi_N(s_1) \leq \Phi_N(t_1)$ and, thus, $\Phi_N(s_1) > \Phi_N(t_1)$.

A further application of SC ensures that $\Phi_N(s_1, t_2) > \Phi_N(t_1, t_2)$. Now, we need to show that $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1, t_2) = \Phi_N(t_1, t_2)$. First, let $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$ and $\Phi_N(s_1) > \Phi_N(t_1)$. However, SC requires that $\Phi_N(s_1) > \Phi_N(t_1) \Rightarrow \Phi_N(s_1, s_2) > \Phi_N(t_1, s_2)$. This is a contradiction. Similarly, it can be shown that $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$ and $\Phi_N(s_1) < \Phi_N(t_1)$ does not hold together. Therefore, $\Phi_N(s_1, s_2) = \Phi_N(t_1, s_2) \Rightarrow \Phi_N(s_1) = \Phi_N(t_1)$. With further application of SC, it can be shown that $\Phi_N(s_1) = \Phi_N(t_1) \Rightarrow \Phi_N(s_1, s_2) = \Phi_N(t_1, s_2)$.

The formulation in (10) is equivalent to the concept of strong separability in the existing literature (Gorman (1968), Blackorby, Primont, and Russel (1978)). Strong separability leads the form of the social welfare index in (2) to be:

$$W(H) = \phi_N\left(\sum_{n=1}^{N} \Delta_{N,n}\left(Q\left(h_{n\cdot}\right)\right)\right) = \phi_N\left(\sum_{n=1}^{N} \cdot_{N,n}\left[U\left(\sum_{d=1}^{D} V_d\left(h_{nd}\right)\right)\right]\right); \tag{11}$$

where ϕ_N is continuous and strictly increasing and $\Delta_{N,n} : \mathbb{R}_{++} \to \mathbb{R}$ is continuous.

According to axiom NM, we require that

$$\phi_N\left(\sum_{n=1}^{N} \cdot_{N,n}\left[U\left(\sum_{d=1}^{D} V_d(\theta)\right)\right]\right) = \theta \text{ for any } \theta \in \mathbb{R}_{++}.$$

In other words, we require $U\left(\sum_{d=1}^{D} V_d(\theta)\right) = \theta$ and $\phi_N\left(\sum_{n=1}^{N} \cdot_{N,n}(\theta)\right) = \theta$. As a result, the form that (11) takes is:

$$W(H) = \Omega_N^{-1}\left(\sum_{n=1}^{N} b_n \Omega_N V^{-1}\left(\sum_{d=1}^{D} a_d V\left(h_{nd}\right)\right)\right), \tag{12}$$

where $\phi_N = \Omega_N^{-1}$, $\cdot_{N,n} = b_n \Omega_N V^{-1}$, $V_d = a_d V$, $a_d, b_n \in \mathbb{R}_+ \; \forall n, d$, $\sum_{d=1}^{D} b_n = 1$, and $\sum_{d=1}^{D} a_d = 1$. Both Ω_N and V are strictly increasing and continuous.

Axiom SP requires each person to be anonymous, and, thus, $b_n = 1/N \; \forall n$. The functional form (12) becomes:

$$W(H) = \Omega_N^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \Omega_N V^{-1}\left(\sum_{d=1}^{D} a_d V\left(h_{nd}\right)\right)\right). \tag{13}$$

Therefore, we express the individual aggregation function as:

$$Q(\cdot) = V^{-1}\left(\sum_{d=1}^{D} a_d V(\cdot)\right)$$

and the standardized achievement aggregation function can be written as:

$$\Phi_N(\cdot) = \Omega_N^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} \Omega_N(\cdot)\right).$$

Following Foster and Székely (2008), axiom LH and axiom PRI result the functional form for the standardized achievement aggregation function to be:

$$\Phi\left(Q\left(h_{1\cdot}\right), \ldots, Q\left(h_{N\cdot}\right)\right) = \begin{cases} \left(\frac{1}{N}\sum_{n=1}^{N} Q\left(h_{n\cdot}\right)^{\beta}\right)^{1/\beta} & \beta \neq 0 \\ \left(\prod_{n=1}^{N} Q\left(h_{n\cdot}\right)\right)^{1/N} & \beta = 0 \end{cases}. \tag{14}$$

Finally, we need to derive the functional form for the individual aggregation function. The individual aggregation function $Q(\cdot)$ is a quasi-linear mean (Eichhorn, 1978, p. 32) since V satisfies CNT and MO. As

$Q(\cdot)$ satisfies NM, Theorem 2.2.1 of Eichhorn (1978) leads the functional form to be:

$$Q(h_{n\cdot}) = \begin{cases} \left(\sum_{d=1}^{D} a_d h_{nd}^{\alpha}\right)^{1/\alpha} & \alpha \neq 0 \\ \prod_{d=1}^{D} h_{nd}^{a_d} & \alpha = 0 \end{cases} \quad \forall n = 1, \ldots, N. \tag{15}$$

Combining (14) and (15) together, we obtain the class of social welfare index in (3). ■

Proof of Theorem 2. At first, we prove the **sufficient** conditions. Let $H_N, H_N' \in \mathcal{H}_N$ for an arbitrary $N \in \mathbb{N}$ and H_N' is obtained from H_N by an association increasing transfer. Let us first consider the situation when $\beta \in \mathbb{R}$ and $\alpha \neq 0$. We summarize parameter α,β, and vector a by θ. We have $\mathcal{W}(H_N;\theta) = \mathcal{F}(F(H_N)) = (F(H_N)/N)^{1/\alpha}$ and $F(\cdot) = \sum_{n=1}^{N} G(\cdot)$.

(i) For $\alpha > 0$ & $\alpha < \beta$, $G(\cdot)$ is L-subadditive. Therefore, $F(H_N') < F(H_N)$ by Proposition 1, which implies $\mathcal{W}(H_N';\theta) < \mathcal{W}(H_N;\theta)$.

For $\alpha < 0$ & $\alpha < \beta$, and $\beta \neq 0$, $G(\cdot)$ is L-superadditive. Therefore, $F(H_N') > F(H_N)$ by Proposition 1, which implies $\mathcal{W}(H_N';\theta) < \mathcal{W}(H_N;\theta)$.

For $\alpha < 0$ & $\beta = 0$, $G(\cdot)$ is L-superadditive. Therefore, $F(H_N') > F(H_N)$ by Proposition 1, which implies $\mathcal{W}(H_N';\theta) < \mathcal{W}(H_N;\theta)$.

Let us consider, now, the situation when $\alpha = 0$ & $\beta > 0$. We have $\mathcal{W}(H_N;\theta) = \mathcal{F}(F(H_N)) = (F(H_N))^{1/n}$ and $F(\cdot) = \prod_{i=1}^{N} G(\cdot)$. In this situation, $\log G(\cdot)$ is L-subadditive. Therefore, $F(H_N') < F(H_N)$ by Corollary 2 and, thus, $\mathcal{W}(H_N';\theta) < \mathcal{W}(H_N;\theta)$.

Hence, if $\alpha < \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SDIA.

(ii) Now, we are going to prove that $\mathcal{W}(\cdot)$ satisfies **SSIA** if $\alpha > \beta$.

First, consider the situation when $\alpha < 0$ & $\alpha > \beta$. In this situation, $G(\cdot)$ is L-subadditive and $F(H_N') < F(H_N)$ by Proposition 1. Thus, $\mathcal{W}(H_N';\theta) > \mathcal{W}(H_N;\theta)$.

Next, consider the situation, when $\alpha > 0$ & $\alpha > \beta$. In this situation, $G(\cdot)$ is L-superadditive and $F(H_N') > F(H_N)$ by Proposition 1. Thus, $\mathcal{W}(H^{N\prime};\theta) > \mathcal{W}(H_N;\theta)$.

In the third situation, we have $\alpha > 0$ & $\beta = 0$. In this situation, $G(\cdot)$ is L-superadditive and $F(H_N') > F(H_N)$ by Proposition 2. Thus, $\mathcal{W}(H_N';\theta) > \mathcal{W}(H_N;\theta)$.

Finally, we have the situation when $\alpha = 0$ & $\beta < 0$. In this situation, $\log G(\cdot)$ is L-superadditive and $F(H') > F(H)$ by Corollary 2. Thus, $\mathcal{W}(H_N';\theta) > \mathcal{W}(H_N;\theta)$.

Hence, for $\alpha > \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SIIA.

(iii) Finally, We are going to prove the third part of the proposition that $\mathcal{W}(H_N;\theta)$ satisfies both WDIA or WIIA when $\beta = \alpha$. First, consider the situation when $\beta = \alpha \neq 0$. We have $\mathcal{W}(H_N;\theta) = \left(\frac{1}{n}F(\cdot)\right)^{1/\alpha}$, but $G(\cdot)$ is a valuation. Therefore, $F(H_N') = F(H_N)$ by Proposition 1 and, thus, $\mathcal{W}(H_N';\theta) = \mathcal{W}(H_N;\theta)$.

Second, consider the situation when $\beta = \alpha = 0$. We have $\mathcal{W}(H_N;\theta) = (F(\cdot))^{1/n}$ and, this time, $\log G(\cdot)$ is a valuation. Therefore, $F(H_N';\theta) = F(H_N;\theta)$ by Proposition 2 and, thus, $\mathcal{W}(H_N';\theta) = \mathcal{W}(H_N;\theta)$.

Hence, for $\alpha = \beta$, $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies both WDIA or WIIA.

Next, we prove the **necessary** conditions. First of all, $\alpha \not< \beta \Rightarrow \alpha > \beta$ or $\alpha = \beta$, which, in turn, implies that the $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SIIA or both WDIA or WIIA, but does not satisfy SDIA.

Secondly, $\alpha \not> \beta \Rightarrow \alpha < \beta$ or $\alpha = \beta$, which in turn implies that the $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SDIA or both WDIA or WIIA, but does not satisfy **SSIA**.

Finally, $\alpha \neq \beta \Rightarrow \alpha > \beta$ or $\alpha < \beta$, which in turn implies that the $\mathcal{W}(H_N;a,\alpha,\beta)$ satisfies SDIA or **SSIA** but does not satisfy both WDIA or WIIA. ■

Proof of Theorem 3. Let H_N',$H_N \in \mathcal{H}_N$ and H_N' is obtained from H_N by common smoothing such that $H_N' = BH_N$. According to Proposition 2, if $\Phi(\cdot)$ is non-decreasing and quasi-concave and $Q(\cdot)$ is concave, then $W(H_N') \geq W(H_N)$. In the formulation of $\mathcal{W}(\cdot)$, $\Phi(\cdot) = \mu_\alpha(\cdot)$ and $Q(\cdot) = \mu_\beta(\cdot)$. From the properties of generalized means, $Q(\cdot)$ is concave if $\beta \leq 1$ and $\Phi(\cdot)$ is quasi-concave if $\alpha \leq 1$. However, for $\alpha = \beta = 1$, $\mathcal{W}(\cdot)$ does not satisfy SICS since $\mathcal{W}(H'_N) = \mathcal{W}(H_N)$. Thus, $\mathcal{W}(\cdot)$ satisfies SICS if $\alpha, \beta \leq 1$ and $\alpha = \beta \neq 1$ and $\mathcal{W}(\cdot)$ satisfies WICS if $\alpha, \beta \leq 1$.

Next, we prove the necessary conditions. First, suppose, $\alpha > 1$. For any $N \in \mathbb{N}$, consider $H_N \in \mathcal{H}_N$ such that $h_{\cdot d} = \mathbf{h} \in \mathbb{R}^N_{++}$ $\forall d$. For every weight vector $a \in \mathbb{R}^D_+$ and for every $\beta \in \mathbb{R}$, the individual aggregation function $\mu_\beta(\cdot; a)$ yields the standardized achievement vector $\mathbf{h}$. Finally, for every $\bar{a} = \mathbf{1}_N/N$, we obtain $\mathcal{W}(H_N; a, \alpha, \beta) = \mu_\alpha(\mathbf{h}; \bar{a})$. Construct an achievement matrix $H'_N = BH_N$, where B is any bistochastic matrix. By construction, $h'_{\cdot d} = \mathbf{h}' \in \mathbb{R}^N_{++}$ $\forall d$. Again, the individual aggregation function yields $\mathbf{h}'$ as the vector of standardized achievements such that $\mathbf{h}' = B\mathbf{h}$. Therefore, $\mathcal{W}(H_N; \alpha, \beta, a, \bar{a}) > \mathcal{W}(H'_N; \alpha, \beta, a, \bar{a})$ and axiom SICS is violated.

Second, suppose $\beta > 1$, $n = d = 2$ and $a = \bar{a} = (0.5, 0.5)$. Let the achievement vectors of the first and the second persons be (h_{11}, h_{12}) and (h_{21}, h_{22}), respectively, such that $h_{11} = h_{22}$ and $h_{12} = h_{21}$. We denote the achievement matrix by H_0. Thus, for every $\alpha \in \mathbb{R}$, $\mathcal{W}(H_0; \alpha, \beta, a, \bar{a}) = (0.5h_{11}^\beta + 0.5h_{12}^\beta)^{1/\beta}$. Construct $\bar{H}_0 = \bar{B}H_0$, where $\bar{B} = \mathbf{1}_{22}/2$. In this situation, for every $\alpha \in \mathbb{R}$, $\mathcal{W}(\bar{H}_0; \alpha, \beta, a, \bar{a}) = 0.5(h_{11} + h_{12})$. If $\beta > 1$, then $\mathcal{W}(H_0; \alpha, \beta, a, \bar{a}) > \mathcal{W}(\bar{H}_0; \alpha, \beta, a, \bar{a})$. Therefore, axiom SICS is violated.

Finally, suppose $\alpha = \beta = 1$. Then $\mathcal{W}(H_N; 1, 1, a, \bar{a}) = \mu(\mu(h_{1\cdot}; a), \ldots, \mu(h_{N\cdot}; a); \bar{a})$ for every $N \in \mathbb{N}$ and for every $H_N \in \mathcal{H}_N$. Construct an achievement matrix $H'_N = BH_N$, where B is any bistochastic matrix. By construction, $\mu(h_{\cdot d}) = \mu(h'_{\cdot d})$ $\forall d$. A little manipulation can show that

$$\mu(\mu(h_{1\cdot}; a), \ldots, \mu(h_{N\cdot}; a); \bar{a}) = \mu(\mu(h_{\cdot 1}; \bar{a}), \ldots, \mu(h_{\cdot D}; \bar{a}); a).$$

Therefore, for every $a \in \mathbb{R}^D_+$

$$\begin{aligned} & \mu(\mu(h_{\cdot 1}), \ldots, \mu(h_{\cdot D}); a) = \mu(\mu(h'_{\cdot 1}), \ldots, \mu(h'_{\cdot D}); a) \\ \Rightarrow\ & \mathcal{W}(H; 1, 1, a, \bar{a}) \not< \mathcal{W}(H'; 1, 1, a, \bar{a}). \end{aligned}$$

Hence, axiom SICS is violated. ■

Appendix B

The generalized mean satisfies the following important properties for every $M \in \mathbb{N}$, for every weight vector $a \in \mathbb{R}^M_+$, and for every $\gamma \in \mathbb{R}$:

(i) (Permutation Invariance) For every $x, x' \in \mathbb{R}^M_{++}$ and for every permutation matrix P such that $x' = xP$,
$$\mu_\gamma(x'; a) = \mu_\gamma(x; a).$$

(ii) (Replication Invariance) For every $x, x' \in \mathbb{R}^M_{++}$ and for every $r \in \mathbb{N}$ such that $x' = [x]_r$,
$$\mu_\gamma(x'; a) = \mu_\gamma(x; a).$$

(iii) (Monotonicity) For every $x, x' \in \mathbb{R}^M_{++}$, if $x' \geq x$ and $x' \neq x$ then
$$\mu_\gamma(x'; a) > \mu_\gamma(x; a).$$

(iv) (Continuity) For every $x \in \mathbb{R}^M_{++}$, $\mu_\gamma(x; a)$ is continuous on $\mathbb{R}^M_{++}$.

(v) (Decomposition) For every $M, M_1, M_2 \in \mathbb{N}$ such that $M = M_1 + M_2$, for every $x \in \mathbb{R}^M_{++}$, $y \in \mathbb{R}^{M_1}_{++}$, and $z \in \mathbb{R}^{M_2}_{++}$ such that $x = (y, z)$, and for every weight vector $a' \in \mathbb{R}^2_+$,
$$\mu_\gamma(x; a) = \mu_\gamma(\mu_\gamma(y; a), \mu_\gamma(z; a); a').$$

(vi) (Concavity) For every $x \in \mathbb{R}^M_{++}$, $\mu_\gamma(x;a)$ is concave (strict) in x if and only if $\gamma \leq 1$ ($\gamma < 1$).

(vii) (Convexity) For every $x \in \mathbb{R}^M_{++}$, $\mu_\gamma(x;a)$ is convex (strict) in x if and only if $\gamma \geq 1$ ($\gamma > 1$)