Department for Education

TALIS Video Study

Case studies of mathematics teaching practices

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DEPARTMENT OF EDUCATION

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1. Introduction

The case studies included in this report have been developed from the videos of mathematics teaching in England that were collected during the Teaching and Learning International Survey (TALIS) Video Study. The TALIS Video Study of mathematics teaching practices is an international study run by the Organisation for Economic Co-operation and Development (OECD), involving eight countries and economies: Biobío, Metropolitana and Valparaíso (Chile), Colombia, England (UK), Germany,[1] Kumagaya, Shizuoka and Toda (Japan), Madrid (Spain), Mexico, and Shanghai (China). It adds to the data provided by the 2018 OECD Teaching and Learning International Survey (TALIS), which asked teachers and headteachers about the teaching and learning conditions in their schools. The TALIS Video Study goes further by going into classrooms and observing teaching and seeking to address questions such as:

- Which aspects of mathematics teaching are related to student learning and non-cognitive outcomes, such as students' self-efficacy in mathematics and their interest in mathematics? And how?
- How do teachers teach in different contexts?
- How are various teaching practices inter-related?

The Department for Education (DfE) commissioned Education Development Trust and the University of Oxford to conduct the TALIS Video Study in England. Data collection took place between October 2017 and October 2018. In the study, researchers gathered videos of two separate lessons from a sample of 85 secondary mathematics teachers in England. These lessons focused on the unit of teaching and learning that includes quadratic equations. The majority of lessons were on solving quadratic equations by using factorisation, by using the quadratic formula, by completing the square, or by finding roots from graphing quadratic functions. In addition, artefacts such as lesson plans, presentations, classroom tasks and homework assignments were collected from these lessons and two other lessons in the unit. The majority of students participating in England were in Year 10, with an average age of 15 years, though the study also included students in Years 8, 9, 10 and 11, depending on which year group was studying the topic of quadratic equations during the period of data collection.

All teachers can benefit from observing other teachers' practice in the classroom and by having their own teaching observed.[2] Many teachers have used peer and/or self-

[1] Germany refers to a convenience sample of volunteer schools
[2] DfE (2014b)

observation as part of their continuing professional development activities,[3] and there are several frameworks available to support such collaborative observations that are specifically focused on mathematics teaching.[4] The TALIS Video Study, together with the case studies highlighted in this report (which are taken from extracts of the videoed lessons), provides a further window for teachers to observe the practice of others. The case studies come from a range of classrooms in a variety of school contexts, and include examples from the full range of class-average prior attainment and socioeconomic background. Each case study was selected because it received the highest possible rating for the teaching practice that it illustrates.

The intention is that the case studies can be used to support excellent mathematics teaching. Excellent – and effective – mathematics teaching includes teachers encouraging reasoning, using rich collaborative tasks, and creating connections between topics both within and beyond mathematics.[5] Students should be encouraged to create, conjecture and experiment, and be given the tools to understand mathematics. This idea is not new,[6] and it has been known for some time that excellent mathematics teachers make purposeful and appropriate use of representations[7] and support students in making connections between mathematical facts, procedures and concepts. The case studies will enable mathematics teachers to see examples of what these practices can look like in the teaching of quadratic equations.

The TALIS Video Study analysed the videos of mathematics teaching against a specially developed framework. This framework focused on six domains or areas of teaching that other research has shown to support students' learning. Four of these domains focus specifically on mathematics teaching, and it is these four that the case studies that follow focus on. These include quality of subject matter, student cognitive engagement, assessment of and responses to student understanding, and discourse. Within each domain, several aspects of mathematics teaching practice, referred to as 'components' in the study, were analysed. In Table 1 (below), the components or aspects of mathematics teaching that were measured within each of the domains are noted in italics, followed by a list of the aspects of each component that the case studies focus on. The table also includes the domains that these components were included within for the study.

The case studies presented in this document focus on illustrations of some of the components or aspects of teaching practices that were deemed valuable in the TALIS Video Study, and which will be of interest to mathematics teachers. Some of these

[3] Jerrim and Sims (2019)
[4] Ingram, Sammons and Lindorff (2018)
[5] NCETM (2008)
[6] Cuoco et al. (1996)
[7] Nunes, Bryant and Watson (2009)

teaching practices are rare in the countries and economies participating in the TALIS Video Study, including England, but are nevertheless highly valued.

Table 1: Components and domains included in the case studies

Domain	**Case Study Component** **(i.e. aspect of teaching practice)**
Quality of subject matter	*Explicit connections* • Connections between equations and graphs • Connections between areas of rectangles and factorising • Connections within contexts • Connections with prior or future knowledge • Connections between methods for solving quadratic equations *Explicit patterns and generalisations* • Looking for and identifying patterns through the sequencing of tasks • Looking for and identifying patterns through the questioning around tasks • Using digital technology to generalise from the mathematics under consideration
Student cognitive engagement	*Engagement in cognitively demanding subject matter* • Using cognitively demanding tasks • Tasks involving multiple approaches

Domain	Case Study Component (i.e. aspect of teaching practice)
Student cognitive engagement (cont.)	*Multiple approaches to and perspectives on reasoning* • Using two methods for one question – and bringing them together All students working with multiple methods – and bringing them together *Understanding of subject matter procedures and processes* • Identifying errors • Visually designating the elements or steps in a process or procedure • Students asking questions
Discourse	*Explanations* • Teacher explanations • Students explaining reasoning *Questioning**
Assessment of and responses to student understanding	*Teacher feedback* • Working with individual students

Note: Components from the TALIS Video Study are identified in this table using italics.
* Questioning is included within the other case studies, rather than having a specific case study that focuses on questioning.

The DfE provides support for teachers to enhance their pedagogy through it's Maths Hubs programme. In 2017/18 academic year, following a successful pilot, Maths Hubs started to roll out Teaching for Mastery programme in secondary schools. Teaching for Mastery focusses on depth of understanding, and encompasses a number of teaching approaches that are aligned with the above domains. For example, it encourages development of procedural fluency and conceptual understanding in tandem, and seeks to provide challenge and the opportunity to deepen understanding for all. At the time of the TALIS video study only a small minority of schools in the sample would have completed their mastery training, however the programme aims to reach 1700 secondary schools by 2023.

Each section of this report includes a summary of some of the wider findings of the TALIS Video Study in relation to the mathematics teaching practices in England. Further details of these findings can be found in the TALIS Video Study National Report[8] and the Policy Report published by OECD[9] entitled *Global Teaching Insights: A Video Study of Teaching*,[10] alongside consideration of the other domains and data collected as part of the study. Pseudonyms are used throughout the report to ensure the anonymity of individual teachers or students that participated in the study.

It should be noted that the case studies are not intended to provide a comprehensive picture of mathematics teaching in England. There are many other valued and highly respected aspects of mathematics teaching which are not captured or included, and many aspects that are difficult to measure in a reliable and valid way.

The TALIS Video Study is an OECD project. The development of the study's instrumentation and data analyses and drafting of international reports were contracted by the OECD to RAND, ETS[11] and DIPF.[12] The authors of this work are solely responsible for its content. The opinions expressed and arguments employed in this work do not necessarily represent the official views of the OECD or its member countries.

[8] Ingram and Lindorff (2020)
[9] OECD (2020)
[10] Also referred to as the TALIS Video Study
[11] Educational Testing Service
[12] Leibniz Institute for Research and Information in Education

2. Making explicit connections within the mathematics

Making connections between mathematical representations, procedures and concepts can help students to develop a rich network of mathematical knowledge. It is not sufficient for teachers to simply include a range of representations, procedures or concepts in their teaching: appropriate connections between them need to be made and used.[13] Mathematics is rich with connections between mathematical concepts, processes and representations.

As explained above, the TALIS Video Study focused on the teaching and learning of quadratic equations. In teaching this topic, teachers will introduce different methods for solving quadratic equations, such as factorisation, using the quadratic formula, and completing the square. Quadratic equations also link to quadratic functions and, in particular, the graphing of quadratic curves. The topic is rich in potential connections that can be made both within and between different representations, as well as between the different methods for solving quadratic equations.

There are a range of mathematical representations (such as equations, graphs and diagrams) between which making connections can support students in developing their knowledge of quadratic equations. There are also connections between the different solution methods commonly taught in England that can support students in deciding which solution method is most effective for different types of quadratic equation. Moreover, quadratic equations can be used in a range of contexts, such as finding the area of rectangles or modelling the trajectory of a ball thrown into the air, which can be helpful in supporting students to use and apply their knowledge of the topic. These contexts can also provide a motivation to solve quadratic equations. Connections can also be made between this and other topics within mathematics, such as linear equations or quadratic functions. This can support the development of a rich network of mathematical knowledge.

In the TALIS Video Study, *explicit connections* are defined as relationships or associations that are clearly stated clearly (verbally or in written form) by a teacher or student. These connections can be between and among ideas, equations, representations, perspectives or procedures, and they can be within the topic of quadratic equations or between quadratic equations and other mathematical topics or real-world settings. Representations can include graphs, tables, diagrams, equations or physical objects. Some representations illustrate mathematical structures that can support students' understanding.

[13] Nunes et al. (2009)

The following case studies illustrate the variety of ways that participating teachers in England made connections when teaching quadratic equations. We present five examples of explicit connections in the remainder of this chapter: connections between equations and graphs; connections between areas of rectangles and factorising; connections within contexts; connections with prior or future knowledge; and connections between methods for solving quadratic equations.

2.1 Connections between equations and graphs

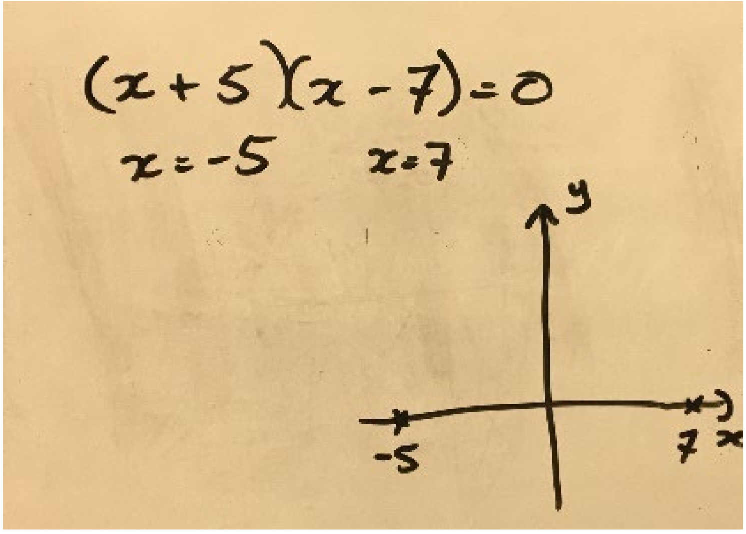
In the lessons from England, the most common connections made were between equations and graphs, and between quadratic equations and quadratic functions. The examples in this section focus on the connections between each of the coefficients or terms within different representations of a quadratic equation and the different features of a graph of an associated quadratic function. In this first example (2.1a) $x^2 - 2x - 35$ is factorised to $(x + 5)(x - 7)$ and the solutions of the equation have been identified as -5 and $+7$ using the rationale that *"we are making each of these brackets equal to zero because zero times anything will give me zero"*.

The teacher shifts the focus to the graph:

Teacher: So okay, what does that tell me about the graph? James?
James: That it will cross the x-axis at -5 and 7.
Teacher: Okay so, my x-intercepts, where it is going to cross my graph are going to be at 7 and negative 5.

The teacher then sketches a pair of axes on the whiteboard, marking in and labelling the two points that have been identified as the x-intercepts, as in Figure 1.

Figure 1: Sketch of a pair of axes with x -intercepts marked



Whilst working on this one quadratic equation, the teacher has described the two values of -5 and $+7$ as solutions, roots, and x-intercepts. The class then continue by identifying the y-intercept from the constant term in the original equation before the teacher summarises:

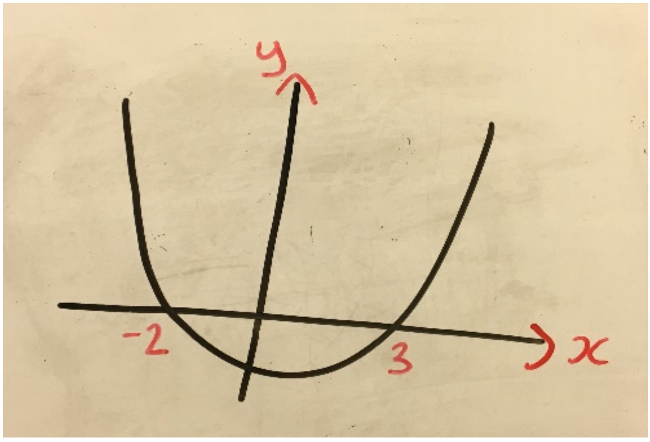
"*So, to see where it is going to cross my x-axis, I did $y = 0$, to see where it is going to cross my y-axis I am going to do $x = 0$*".

By the end of this discussion, the class has identified the roots from the factorised form of the equation and has made the connection that these roots are the same as the x-intercepts on the graph. The students have then made the connection between the constant term in the original equation and the y-intercept.

[Taken from Lesson 2 with a Year 10 class]

The connections between the equations and the graph can also be made in the opposite direction. In this second example (2.1b), the teacher has sketched a parabola on the board and has marked 3 and -2 as the x-intercepts (see Figure 2). The teacher then asks, "*what equation would that graph represent?"* The student gives the expression $(x + 2)(x - 3)$, which is in factorised form. The teacher then highlights the differences between the expression given and the equation of the quadratic curve, explaining: "*that expression goes with that quadratic curve. The curve, if we were looking at the curve, it would be $y =$. If we are looking at the equation, it would be $= 0$.*"

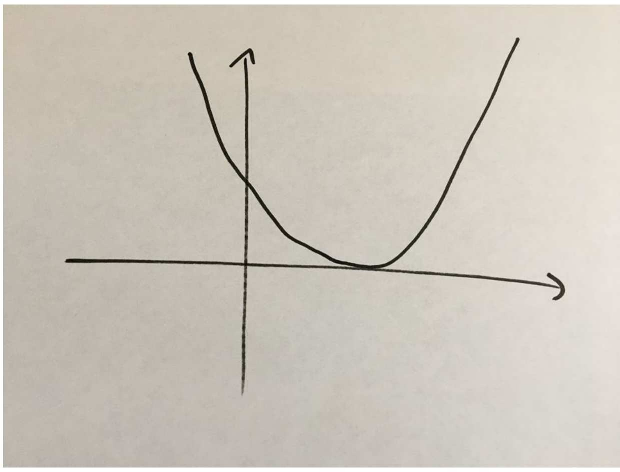
Figure 2: Sketch of a parabola with 3 and -2 marked as the x-intercepts[14]



This connection between a quadratic equation and the graph of the related quadratic curve becomes particularly useful when considering repeated roots or cases where the equation has no real solutions. In this example, the student focuses on a quadratic function with a positive x^2 coefficient.

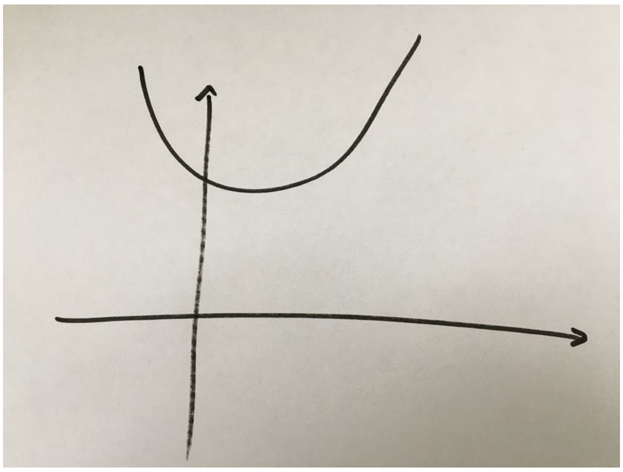
Teacher: What would one repeated root look like?
Ariel: Would it not go below the x-axis?
Teacher: It wouldn't go below the x-axis. It would just touch the x-axis at one point [as in Figure 3]. So, Kristine, I wonder what happens when there's no solutions?
Kristine: It doesn't touch the x-axis at all.
Teacher: It doesn't touch the x-axis at all [as in Figure 4].

Figure 3: Sketch of a parabola that touches the x-axis



[14] There are several quadratic equations whose graph could be represented by the diagram in Figure 2 but the teacher focused on one particular equation where the coefficient of the quadratic term was 1.

Figure 4: Sketch of a parabola that does not cross or touch the x-axis.



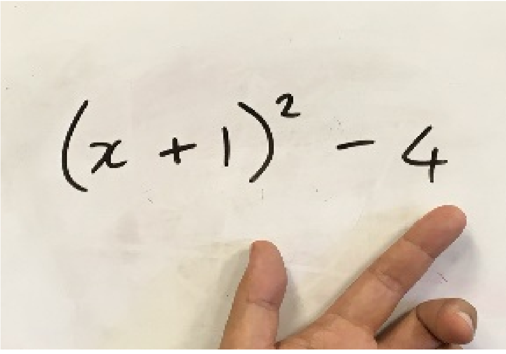
[Taken from Lesson 1 with a Year 10 class]

When equations are given in completed square form, a connection can also be made to the turning point of the parabola. In the third example (2.1c), the class have been drawing the graph of an equation from a table of values. They have subsequently rearranged the equation $y = x^2 + 2x - 3$ into completed square form, $y = (x + 1)^2 - 4$. The teacher then helps students to make the connection between the turning point of the graph they have drawn and the completed square form of the equation of that same graph (as detailed below).

Figure 5: Teacher pointing to the turning point of the function

Teacher: What does x have to be to make $(x + 1)$ equal to zero?
Students: Minus one.
Teacher: Minus one. What's the x-coordinate of the turning point here [pointing to the graph as in Figure 5]? Minus one…
Ash: Minus 4.
Teacher: Look, minus 1 minus 4 [pointing to the expression as in Figure 6].

Figure 6: Teacher pointing to the two numbers in the expression



The class working together go on to identify the turning points of another equation for which they have already drawn the graph, and found the completed square form of: $y = (x - 2.5)^2 - 0.25$.

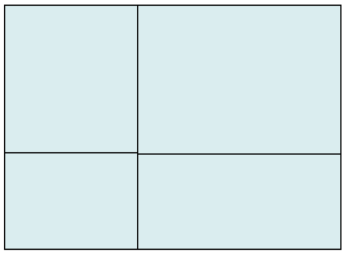
[Taken from Lesson 2 with a Year 10 class]

2.2 Connections between areas of rectangles and factorising

Another connection that is frequently made in the teaching of quadratic equations is between the areas of rectangles and the process of factorising. In the task (example 2.2a) presented in Figure 7, the teacher initially asks the students to discuss (in pairs) the two questions at the bottom of the slide. In the discussion that follows, the sides of the rectangle are labelled and the teacher emphasises that the square must have sides t and t (because it is a square, not a rectangle, meaning its length and width must be the same). The conclusion is reached that the two algebraic expressions represent the area of the large rectangle. The teacher then asks students why $t^2 + 6t + 4t + 24$ and $t^2 + 10t + 24$ are also expressions for the area of the large rectangle, as in the discussion below.

Figure 7: Task connecting the area of rectangles to quadratic equations

A rectangle has a length of $t + 6$ and a width of $t + 4$



Why can the area be expressed as $(t + 6)(t + 4)$?

Why can the area be expressed as $(t + 4)(t + 6)$?

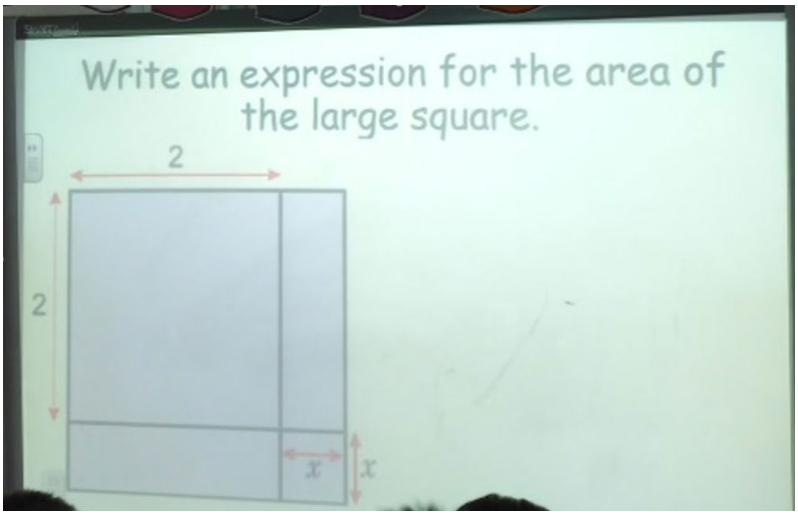
Teacher: Okay then, I've got slightly different looking expressions now, but for the same thing. Where is this first one going to come from, $t^2 + 6t + 4t + 24$? Go on, Sam?
Sam: t times t is t^2. t times 4 equals $4t$. 6 times t equals $6t$ and 4 times 6 is 24.
Teacher: Exactly. So, I am multiplying together, because I'm thinking about area – that's why I am multiplying my length by my width. And, as Sam said, I've got these four regions in there.
Teacher: Ashley, I heard you kind of very succinctly say why the second one is the same as the first one.
Ashley: Because it is just simplified.
Teacher: Exactly, it is just a simplified version.

Throughout, the teacher has emphasised the relationship between the area of the rectangle and the algebraic expressions. This continues as she asks the students to make the connections in the other direction, starting with the expression $w^2 + 5w + 8w + 40$, and stating that this is the area of the rectangle before asking the students to identify the length and the width. The students do this by identifying that the square must have area w^2, and therefore the lengths of the sides of that square must therefore be w. The students then identify which rectangle must have an area of 40 ("*because it can't be where the w is*"). They go on to identify the lengths of this rectangle as 5 and 8 because of the coefficients of the *w* terms in the expression. The teacher moves on to an expression that has already been simplified, $w^2 + 10w + 21$, and asks the students to work out which lengths of the rectangle with an area of 21 would result in 10 as the coefficient of the w term.

[Taken from Lesson 1 with a Year 10 class]

Similarly, the connection between the area of squares and factorising can be used to help students build towards completing the square. In the activity discussed in this example (2.2b), shown in Figure 8, the teacher begins by asking students to write an expression for the area of the large square on their mini-whiteboards.

Figure 8: Task asking students to write an expression for the area of a square



Initially, all the students give an answer of $(2+x)(2+x)$. The teacher then asks the class to expand this, and after some discussion, the students reach the expressions $x^2 + 4x + 4$ or $4 + 4x + x^2$. The discussion continues as follows:

Teacher: But going back to my original question, what is a nicer way of writing that? Nicky?
Nicky: Just […] put a square around the first bracket.
Teacher: Good, so I can do $(x+2)^2$, can't I? So, this is what we call a perfect square. Write that in your books please. Expressions such as $(x+2)^2$ are known as a perfect square.
The class then continues to work on finding the area of a variety of squares.

[Taken from Lesson 1 with a Year 9 class]

These examples of connections between areas of rectangles and factorising also illustrate the use of cognitively demanding tasks in mathematics teaching, discussed further in section 4.1.

2.3 Connections within contexts

Connections can also be made between problems set in different contexts, and the quadratic equations that might be used to solve these problems. These contexts could be mathematics-based, such as problems involving areas and perimeters of rectangles, or could be from real-world situations, such as solving problems involving a ball being thrown.

In this first example (2.3a), the teacher has posed the following problem (see Figure 9):

Figure 9: Problem about the area of a rectangle

A rectangle has a length that is 8cm more than its width. The area of the rectangle is $20cm^2$. Find the dimensions of the rectangle.

The students work on this problem on their mini whiteboards, and the following discussion takes place:

Teacher: It says: "the length is 8cm more than the width," so what could I name these sides?

Peter: a and b

Teacher: I'm not going to name them a and b, and I'll come back to that in a second. Alex.

Alex: x

Teacher: Good. If I have x and I have $x + 8$, that means that my width is x and my length is my width plus 8. And it says that my length is 8 more than my width, so that is right, yeah?

Teacher: Peter suggested that we call them a and b. Why do I not want to call them a and b? Sara?

Sara: Because you want x to be the same value and a and b will be separate values.

Teacher: Good! Because how many unknowns can I have in one equation? I can only solve a singular equation if I have one unknown. If I have two unknowns, what do I need to do? What kind of equations can I solve if I've got two unknowns? Avery?

Avery: Simultaneous.

Teacher: Yeah, simultaneous, good. But I am not doing that right now. So, I've got x and $x + 8$. How do I find the area? What do I do to those two sides? Parker?

Here, the teacher has sketched an image of the rectangle, and the students have labelled the sides x and $x + 8$. The teacher also makes a brief connection here to simultaneous equations, highlighting that two simultaneous equations are needed to calculate the value of two unknowns. This is a connection with prior knowledge (see section below). As the class generates an equation for the quadratic equation, the teacher continues to make connections between the area of the rectangle and the quadratic equation formed.

The context becomes important again once the students have identified the two solutions of the quadratic equation they have formed. The teacher continued the discussion as follows:

Teacher: At this point, I have two solutions: I've got $x = 2$ and $x = -10$. What can I immediately decide? What should go through your mind? I am doing this problem and I've got $x = 2$ and $x = -10$. Jordan?

Jordan: Hmm, that the answer is 2 because you can't have a –10 length.

Teacher: Perfect. So, my two solutions are $x = 2$ and $x = -10$ but my only possible solution is $x = 2$, because I can't have a length here of –10, can I? 2 is my only option. So, my length is 2.

As indicated above, in the context of the areas and lengths of rectangles, it is not possible to have a negative length. The students identify this feature of the context in order to conclude that the final solution is $x = 2$.

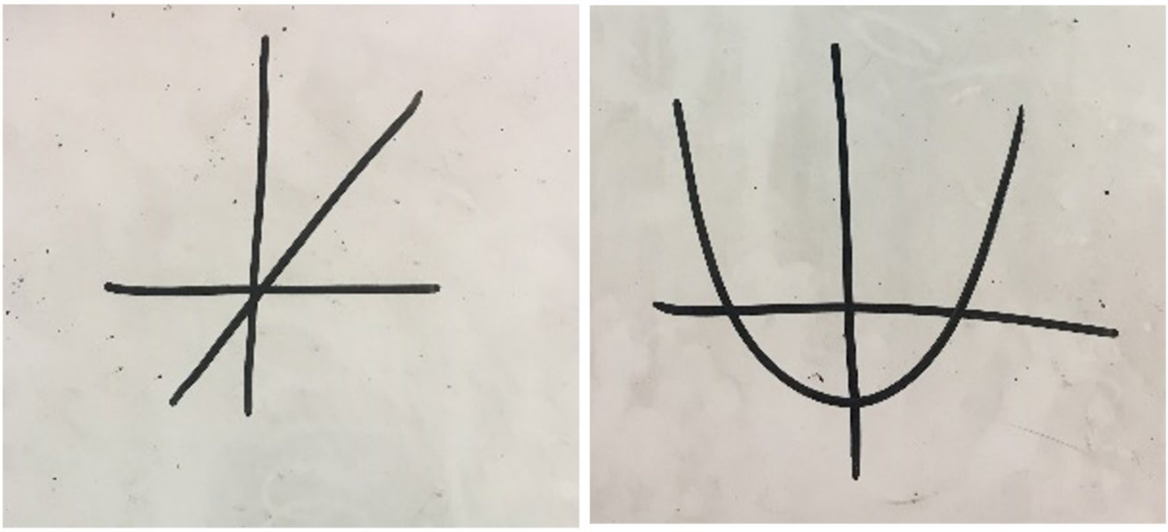
[Taken from Lesson 2 with a Year 10 class]

2.4 Connections with prior or future knowledge

Connections can also be made between quadratic and linear equations, and in this study, the most common connection made was between linear graphs and quadratic graphs. In one videoed lesson (example 2.4a), the teacher uses the transition from working with linear graphs to quadratic graphs to address a question raised by one of the students:

Riley: I don't understand why you would want two solutions.
Teacher: Why do I want two solutions?
Riley: Yeah, like why does the formula give you two solutions instead of one?
Teacher: Well, there are two questions there. 'Why does the quadratic give me two solutions?' is one question. If I think about the linear graph, what does the linear graph look like? A straight-line graph, linear graph.
Riley: Yeah, straight line.
Teacher: So, if I have a linear graph, I am just going to have a straight line like this [sketches a linear graph the board as in Figure 10].
If I am looking at a quadratic graph, what does a quadratic look like? Yes there we go, Elliott has just done it… this or this [sketches a parabola first in the air and then on the board as in Figure 10]. How many times does this quadratic cross my x-axis?
Students: Two.
Teacher: So why would we have two solutions? Because this is what we are going to find when we come on to doing quadratic graphs, this [points at the two x-intercepts on the sketch of a parabola] is what we are going to find. That is why it gives you two solutions, that is the first question.
The second question is why does the formula give us two solutions? And that is because we have + or – in our formula. And therefore, we have one value where we are using the positive sign and one where we are using the negative.

Figure 10: Sketches of a linear function and a quadratic function



Here, the teacher has made a connection between the equations ($y = x^2 + 5x + 4$ and $(x+1)(x+4) = 0$), the graph, and the formula $x = \frac{-5 \pm \sqrt{5^2 - 16}}{2}$

[Taken from Lesson 1 with a Year 10 class]

At GCSE level, most students are taught that quadratic equations have two solutions, one solution or no solutions. Some teachers use students' encounters with quadratic equations with different numbers of solutions to introduce the idea of repeated roots and no real solutions, meaning that there are solutions, but these solutions involve complex numbers and consequently have imaginary roots. This is made explicit by a teacher in the example below (2.4b).

Teacher: So, [for] quadratics you can have two solutions. And these solutions are sometimes called 'roots'. 'Roots' is the word we tend to use at A-level. Sometimes they have one repeated root, sometimes they have no solutions. Right, well there are solutions, but they are called imaginary numbers. If you want to do A-level Maths and if you want to do Further Maths, you learn about imaginary numbers.

[Taken from Lesson 1 with a Year 10 class]

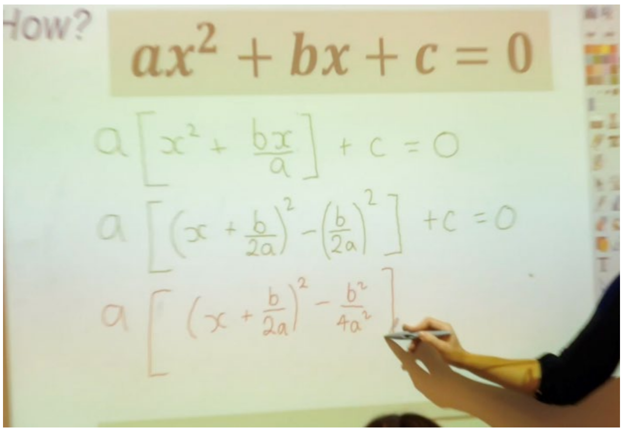
2.5 Connections between methods for solving quadratic equations

There are many methods and approaches students can use when working with quadratic equations. Many of these are illustrated later in the section on student engagement in cognitively demanding subject matter, in the case studies on using multiple approaches to and perspectives on reasoning. Typically, three particular methods are taught in

England: factorising, using the quadratic formula, and completing the square. Another way in which teachers make explicit connections is by making connections between these methods.

In the example which follows (2.5a), the students are working with the general form for a quadratic equation and are completing the square.

Figure 11: The first three steps used to derive the quadratic formula.



The class work through the first three steps together (see Figure 11), before working individually. The students are given the option of working on a sheet on which the steps are shown, but on which the student needs to explain the steps taken. At the end of the process, the students reach the quadratic formula. The teacher has thus enabled the students to experience the connection between completing the square and the quadratic formula, by working through a '*proof of the quadratic formula'.*

[Taken from Lesson 2 with a Year 10 class]

2.6 Summary

Explicit connections between mathematical concepts, process and procedures were not commonly used in the lessons from England within the unit that included quadratic equations. Connections to other mathematical topics such as linear equations, were also rare. In contrast, connections between different representations of the quadratic equations were common, particularly connections between the equations of quadratic functions and their graphs.

All students need access to the key mathematical concepts and big ideas within the National Curriculum,[15] and to the rich connections between them. Mathematics is a coherent discipline and it is by examining and unfolding the connections within and

[15] DfE (2014a)

between concepts that students come to understand mathematical ideas. Interestingly, in the analysis of the lesson videos from the TALIS Video Study[16], there was a significant relationship between the average socio-economic status of students in a given classroom and the quality and quantity of explicit connections that teachers used. Classes with higher average levels socio-economic status experienced lessons where explicit connections were made more frequently than classes with lower average levels.

Connections between representations also help expose the mathematical structure being taught. Making connections between different approaches, processes or procedures, and examining the similarities and differences between them, can also support students in identifying the underlying structures within the mathematics. In the case of quadratic equations, there are different algebraic representations of the same structure – for example, $30x^2 - 28x + 6$ is equivalent to $(5x - 3)(6x - 2)$ – which offer "different interpretations of the same structure".[17] Making connections between different contexts can also help students to see the similar mathematics in these different contexts. By paying attention to the underlying mathematical structures shared between concepts, process and representations, teachers help students to make connections between them.

[16] Ingram and Lindorff (2020)
[17] Hoch and Dreyfus (2004). p51.

3. Teachers and students using explicit patterns and generalisations

Problem-solving in mathematics involves finding and using patterns to make conjectures that lead to generalisations. Key aspects of mathematical reasoning include recognising and analysing patterns, and articulating structures that lead to these mathematical generalisations. Processes and procedures themselves involve patterns of actions such as rearranging equations, factorising, or equating, and it is these patterns that reveal the underlying mathematical structures that support students' understanding.

Patterns and generalisations underpin many mathematical concepts, procedures and processes, and the case studies which follow illustrate how teachers or students looked for or identified patterns, or generalised from the mathematics under consideration. For the purposes of the study, a pattern was defined as "an ordered set of mathematical objects", such as numbers, equations, graphs or problems, or "a recurring sequence". For generalisation, the definition was taken from the work of Kaput who describes generalisation as follows:

> *"Generalization [sic.] involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, higher level objects of reasoning or communication)."*[18]

When analysing the data for the study, to count as an example of using patterns and generalisations there needed to be at least two instances referred to or investigated during the lesson from which a generalisation or pattern was developed. The study also makes a distinction between whether it is the teacher who is looking for patterns, identifying patterns, or is making explicit generalisations, or whether it is the students who are doing this. In many of the examples included here, the students themselves are working with patterns and generalisations in some way. In the sections that follow, a distinction is made between examples where the students are working directly with the patterns in some way, but it is the teacher that makes the generalisation, and examples where the students make the generalisation themselves from these patterns.

[18] Kaput (1999). p136.

The case studies on explicit patterns and generalisation are divided into three categories that focus on the tasks, questions and tools teachers used to offer opportunities to see or use patterns or generalisation in their learning of quadratic equations. These are: looking for and identifying patterns through the sequencing of tasks; looking for and identifying patterns through the questioning around tasks; and using digital technology to generalise from the mathematics under consideration.

3.1 Looking for and identifying patterns through the sequencing of tasks

One way of supporting students to work explicitly with patterns is to sequence a series of tasks that will enable them to look for patterns within the sequence. In this first example (3.1a), shown in Figure 12, the teacher uses a carefully constructed sequence of four quadratic expressions, which allows students to look for patterns without being instructed as to which patterns to look for.

Figure 12: Sequencing a series of tasks[19]

Expand the following expressions:

$$(x+1)^2$$

$$(x+2)^2$$

$$(x+3)^2$$

$$(x+4)^2$$

Think about any patterns that you can see.

After the students have spent some time working individually on this sequence, discussing their patterns within the group around their table, the teacher brings the class together to discuss the questions:

Teacher: But really, we are more interested in these patterns. So, there is really no silly answers to this at all, any patterns that you can see going on there? Ali?

Ali: It looks a bit like x^2 and then in the next x added by itself is the middle number, so like 1 and 1 is 2, and then the last number is the number squared.

Teacher: The number squared, okay. So, that middle column, those coefficients of x, tend to be, if I'd not simplified it, tends to be like $x + x$ or $2x + 2x$, so I've got two lots of them. Brook, go on.

[19] Reprinted from integralmaths.org with permission from MEI

Brook: So if you've like got x for the first thing and $+1$ or $+2$ for the second thing, you would power the first thing and then the second thing you would [inaudible] and then 2 times the first thing times the second thing.

Teacher: Okay, right, you pretty much summed it all up together there. Let's have a look. We've got the square of whatever this number is. Okay? We've got the square at the end there, all the square numbers? That would carry on forever, wouldn't it? And that coefficient of this x there, that linear term, okay, we can see, is always two times as big as each of the numbers at the end of my bracket.

In this example, the open nature of the original task allows the students to notice both the relationship between the numbers inside the bracket and the constant term of the expanded form, as well as the relationship between the numbers inside the bracket and the coefficient of the x term in the expanded form. The teacher then invites some students to share the patterns they have noticed before summarising the relationships for the whole class.

[Taken from Lesson 2 with a Year 10 class]

A similar example is given in Figure 13, in which the sequence of tasks draws students' attention to the pattern relating the numbers in the factorised and expanded form of quadratic expression (example 3.1b).

Figure 13: Sequencing questions on factorising and expanding quadratic expressions

Expand	Factorise
• $(x+2)(x+3)$	• $x^2+8x+12$
• $(x-2)(x+3)$	• $x^2+7x+12$
• $(x+2)(x-3)$	• $x^2+13x+12$
• $(x-2)(x-3)$	• $x^2-13x+12$
	• $x^2+4x-12$
	• x^2+x-12
	• $x^2+11x-12$

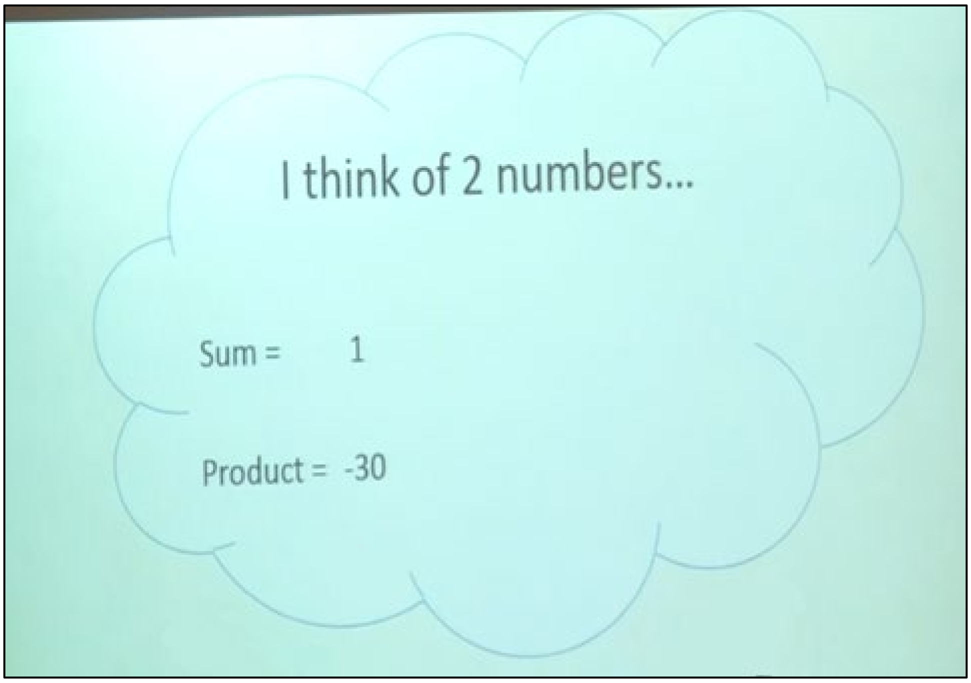
[Taken from Lesson 1 with a Year 10 class]

The four questions in the section on expanding brackets vary only the sign of the numbers inside the brackets. This offers students the opportunity to see the impact this has on the coefficient of the x term and on the constant term. Similarly, keeping the magnitude of the constant term the same in the sequence of questions about factorising offers students the opportunity to focus on which factors of the constant term sum to form the x coefficient, and therefore the features of these factors which affect the x coefficient.

3.2 Looking for and identifying patterns through the questioning around tasks

Another way to use a sequence of tasks is where the pattern is not visible through the sequence itself, but becomes apparent through the teacher's questioning around the sequence. In this next example (3.2a), the students have been given the sum and product of two numbers, and need to work out what the two numbers were, as in Figure 14. Initially, the questions only involve the natural numbers, but after a few examples, the teacher introduces negative products or sums.

Figure 14: Finding two numbers when given the sum and product

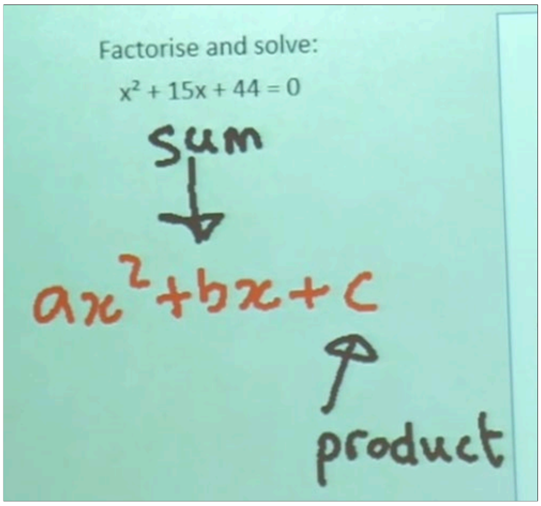


The teacher then introduces the equation $x^2 + 15x + 44 = 0$ on the whiteboard and asks:

Teacher: Which one is the product, which one is the sum? Which one? Think back to your generic formula of $ax^2 + bx + c$. Which one represents the product of the two numbers, which one represents the sum of the two numbers when I expand the brackets?

The class works through this specific example to identify the sum as 15 and the product as 44, leading to $(x + 11)(x + 4)$. The teacher writes up the general form of a quadratic expression and labels the coefficient of x as the sum and the constant term as the product (see Figure 15 – note this 'rule' is only true when $a = 1$).

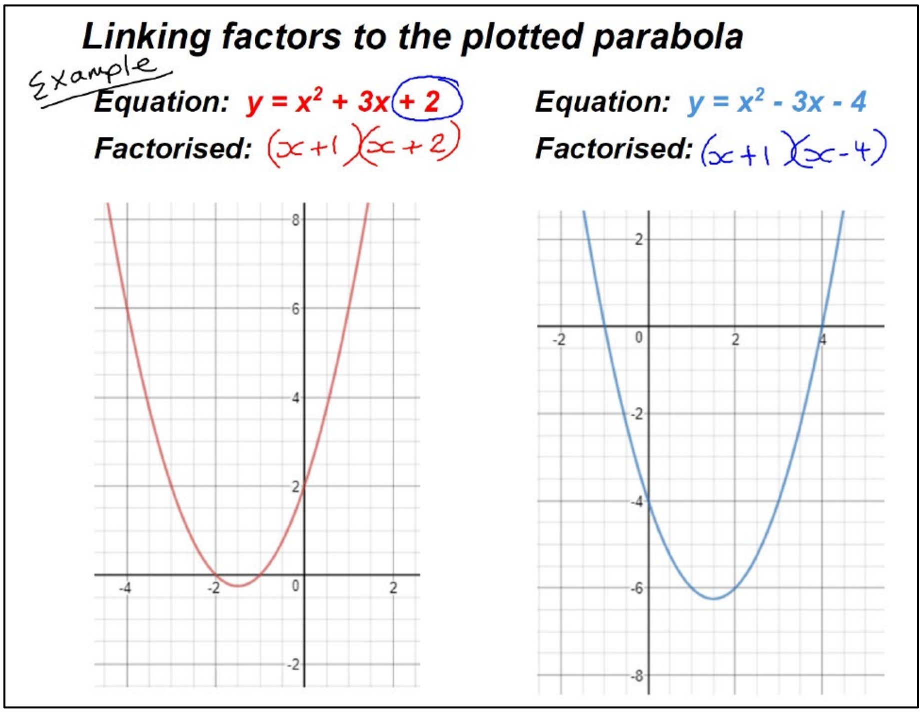
Figure 15: A general rule for interpreting the quadratic expressions



[Taken from Lesson 1 with a Year 9 class]

This strategy is also used by teachers to connect algebraic and graphical representations of quadratic equations. In this next example (3.2b), the students are given two equations which are presented in three different representations – expanded form, factorised form, and in a graphical form – and are asked to find a link between the factors in the factorised form and the graphs (see Figure 16).

Figure 16: Linking factors to parabolas



Initially, a couple of students focus on the relationship between the expanded form and the graph:

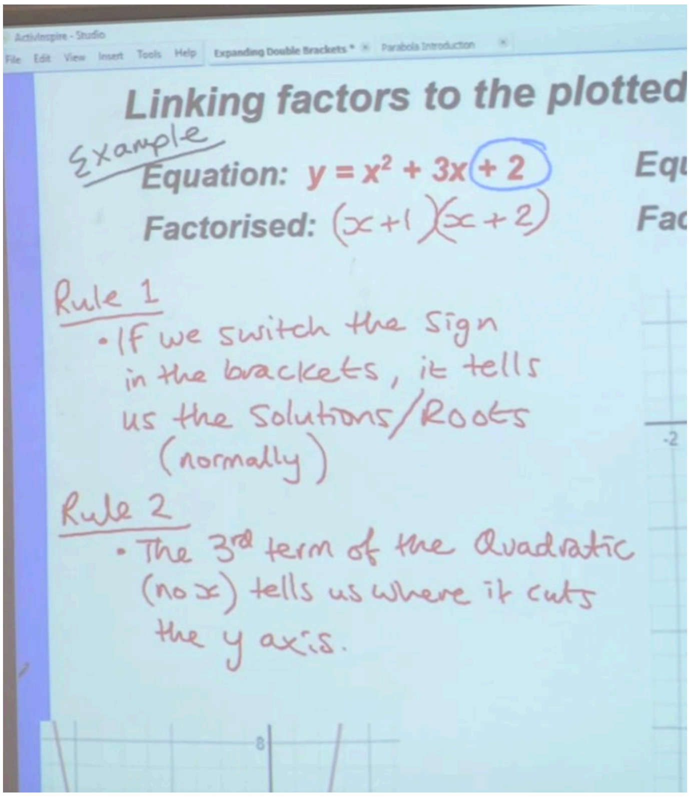
Teacher: Alright Corey, you guide me through what you think. Guys, have a listen then.
Corey: I think the last number in the bracket, in the second bracket.
Teacher: The last number in the second bracket. That one? And that one? [points on board]
Corey: Yes, that's where it crosses through the y-axis.
Teacher: Ahh, okay, I see that. So, the +2 is the +2 and the –4 is the –4 [points on board]. I hadn't noticed that, but you spotted it, well done. That wasn't actually the link but very clever. Maybe I should have done two different examples. It does work, but it is not the link I was looking for.

Later, the discussion shifts to the factorised form:

Toni: Oh, it cuts on the x before it cuts on the y.
Teacher: Hang on, here?
Toni: Yeah on that one? On the right. It cuts on the 4 and the 1.
Teacher: Alright, it does cut through at 4. What does that have to do with my factorised brackets?
Toni: Well you got -4 and on the left you got -1, which …
Teacher: Alright, so what's wrong with the numbers in here?
Wynne: They are the wrong way.
Teacher: It is not so much the wrong way. Keep going Toni?
Toni: They are like the opposite sign.
Teacher: They are the opposite signs. So, let's just see if that works. If I was to swap the sign to make this a positive 4, it creates one of what we call the solutions. So, by swapping that to a positive 4, it cuts here. By swapping that sign, I am going to get a negative 1 and it cuts here. Let's see if it works on the red. If I swap *a* $+2$ by changing the sign, that would be *a*?
Students: -2.
Teacher: -2. Does it cut at -2?
Students: Yes.
Teacher: Aha. And that one?
Harper: It would make -1.
Teacher: And does it cut at -1?
Students: Yes.
Teacher: Right, so that is a very strong pattern and we are going to write that up in just a second.

In this example, the students have worked on two specific examples to identify a relationship between the algebraic representation and the graphical representation, which the teacher then summarises as 'rules' (using the language of the students), as shown in Figure 17. These two rules summarise the patterns that the students noticed and offer the opportunity for these patterns to be refined and challenged as the class consider more examples and counterexamples.

Figure 17: Rules connecting factors to plotted parabolas



[Taken from Lesson 2 with a Year 8 class]

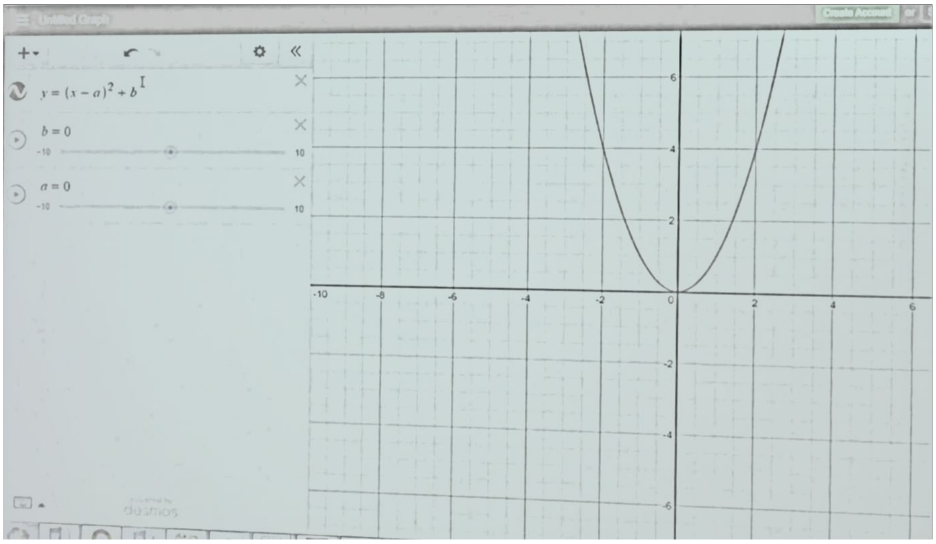
3.3 Using digital technology to generalise from the mathematics under consideration

Another way in which teachers can support students in noticing and identifying patterns is through the use of digital technology, for example, by using dynamic geometry software.

In the next example (3.3a), the class have been working together on the specific equation of $y = x^2 + 4x + 4$, which they have factorised and solved to find the repeated root, which is then connected to the x-intercept on the graph of the equation. The class moves on to working with $y = x^2 + 7x + 10$ in the same way. The teacher then asks "what does completing the square do for us?", before introducing a graph with the equation in completed square form ($y = (x - a)^2 + b$), where the values of a and b can be altered using sliders (as shown in Figure 18).

Teacher: So, this is our completed square form and this is quite interesting, we'll look at this a little more when we start transforming graphs, but if we take the number on the end, the +b, if we change that, we get that movement in the y-direction, and if we change the a, our curve moves left and right [shows on screen]. And if I leave it there, when $b = 1$ and $a = 2$, you can see where it sits. Now, were does the 1 and the 2 come in? If I were to write that out as an equation, that would be $y = (x - 2)^2 + 1$

Figure 18: Using dynamic graphing software to see the effect of changing the values of the variables



Teacher: Now, so what do you think the significance of those numbers is? Because when we think about factorising quadratics, that helps us see where the line cuts the x-axis, what is the significance of these two numbers for this graph? Can anyone spot anything there?
What if I change the *a* to 3? [changes value of *a* using the slider]? What if I change it to maybe -1? So, what is the significance of this number in the bracket? Can anyone spot where it might have come from?

Over the next few minutes, a range of students make suggestions about the effects of changing a and b, finally concluding that the values related to the coordinates of '*the lowest point'*, which the teacher names as '*the minimum point'*.

[Taken from Lesson 2 with a Year 10 class]

Digital technology allows teachers and students to work with a large number of examples in a short period of time. In particular, by keeping some parts of the equation the same whilst varying others, patterns and structures within the examples can emerge that expose the relationship between the algebraic representations of the quadratic equations and the graphs of the quadratic functions. Newer technologies are now widely available for supporting the teaching and learning of algebra, such as graph-plotters, dynamic geometry apps and spreadsheets.[20]

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[20] Godwin and Sutherland (2007)

3.4 Summary

The use of explicit patterns and generalisations within the unit that included quadratic equations was not very common in the lessons in England. In the majority of cases, where patterns and generalisations were used, it was the teacher who used the pattern to reach a generalisation, rather than the students.

One way in which students come to understand mathematics as a coherent discipline is by considering small connected steps or representations that can then lead to a generalisation of a concept. Consequently, they will be more able to apply concepts in a range of contexts, or know when not to apply a concept.[21] Teachers can use carefully constructed sequences of patterned examples to highlight a generalisation, or they can use unsequenced examples and draw attention to the patterns through questioning and then make generalisations. Students need opportunities to describe generalisations based on mathematical structures and inductive reasoning from sequences of examples or approaches.[22] Teachers can also use digital technology to expose patterns and invite students to draw generalisations.

Algebra in particular expresses generalisations and relationships between different representations, such as symbols and graphs.[23] The topic of quadratic equations involves several types of transformational activities[24] (rule-based activities), such as factorising, expanding brackets, simplifying expressions and solving equations. Generational activities[25] (generating expressions and equations that are the objects of algebra, expressions of generality and rules) that can arise through the use of patterns or sequences of examples were less common in the lessons in the TALIS Video Study.

[21] Mason (2008)
[22] Nunes, Bryant and Watson (2009)
[23] Kieren (2004)
[24] Kieren (2004)
[25] Kieren (2004)

4. Ways of offering opportunities for students to engage in cognitively demanding subject matter

A topic like quadratic equations involves students working extensively with procedures and practicing algebraic manipulation. Many teachers also include tasks or activities which offer students opportunities to engage in cognitively demanding subject matter, such as analysing, creating or evaluating work that is cognitively rich and requires thoughtfulness. Examples include detailed examinations or explorations of the features and relationships among mathematical procedures, processes or ideas, formulating or inventing a way to solve a problem, and determining the significance or conditions of a mathematical idea, topic, representation or process. These tasks and activities go beyond recall or the rote application of procedures. The cognitive demand of the subject matter is based on how students are being asked to engage with the mathematics, rather than the difficulty of the mathematics itself.

The case studies focus particularly on *the tasks teachers use* to offer students these opportunities and illustrate the ways in which teachers can work with these tasks. The first section, on cognitively demanding tasks, focuses on those tasks in which students engaged in analytical and creative processes as they worked on the task. The second section, on tasks involving multiple approaches, focuses more on tasks in which students used different approaches to the same problem and these approaches are shared and discussed with the class as a whole.

4.1 Cognitively demanding tasks

The first example (4.1a) is of a task in which students need to create equivalent expressions, emphasising the equivalence of different algebraic representations. This is shown in Figure 19.

Figure 19: An example of a cognitively demanding task

STARTER

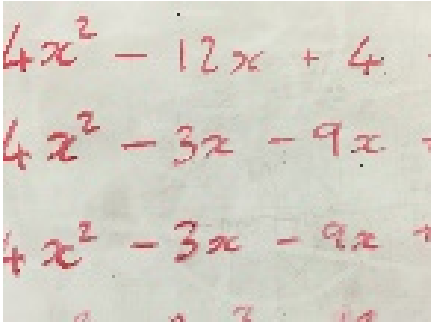
On a MWB, write some algebraic expressions which are equivalent to

$4x^2 - 12x + 9$

After a few minutes of students writing a range of equivalent expressions on their mini whiteboards, the teacher draws the class together to discuss the equivalency of these expressions:

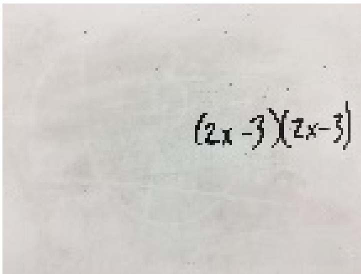
Teacher: Okay, Daniela, can I have your board please? Any of these expressions, please have a look, where she has got $-3x - 9x$, they are equivalent. They have not been simplified but they are equivalent. [Returns board].

Figure 20: Daniela’s board of equivalent expressions



Teacher: Can I have a look at these two? [Picks them up.] Okay, so I got these two here. This one is yours Liz? What did you do to get that?

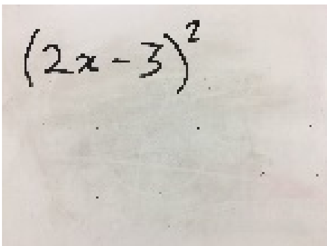
Figure 21: Liz's board with an equivalent expression



Liz: I factorised.

Teacher: Lovely, so she’s factorised this expression and she ended up with this. $(2x - 3)(2x - 3)$. And Peter ended up with this one. Are they the same? How come?

Figure 22: Peter's board with an equivalent expression



John: This is just times together, which is the same.

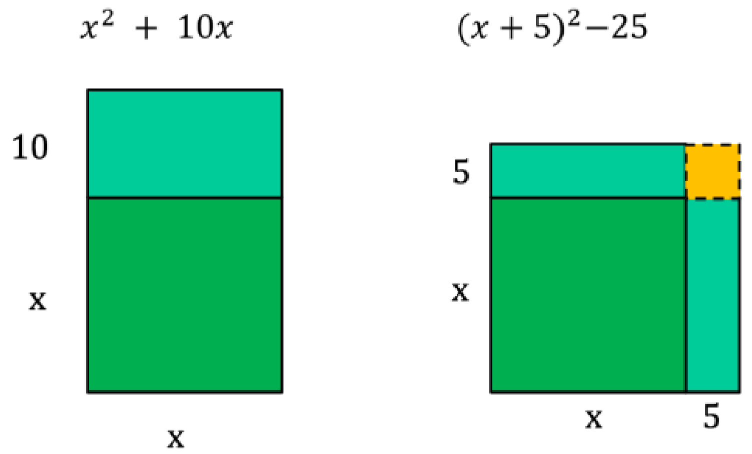
Teacher: Exactly, we could write – this is important, pay attention to this – we could write $(2x - 3)(2x - 3)$ as $(2x - 3)^2$ Yeah? So, these things are the same, fab.

[Taken from Lesson 2 with a Year 10 class]

In this example (4.1a), the equivalence between the factorised form and the expanded form is being emphasised, rather than the process of factorising. The students have created equivalent expressions, which involves some thoughtfulness.

In this second example (4.1b), the class have been completing the square on expressions of the form $x^2 + ax$, using images of rectangles and squares to support them (as shown in Figure 23).

Figure 23: Using areas of rectangles to learn about completing the square



The teacher then invites the student to consider the case of $x^2 + 6x - 120$, where the expression has three terms, instead of the two that the class has been working with so far. Students are therefore formulating their own method for completing the square for expressions of this type. One student begins by suggesting that the expression will need to include $(x + 3)^2$. The teacher then asks another student to expand this expression to get $x^2 + 6x + 9$. The teacher highlights that they are looking for $x^2 + 6x - 120$ and have $x^2 + 6x + 9$. Another student suggests that 129 needs to be subtracted to get $(x + 3)^2 - 129$.

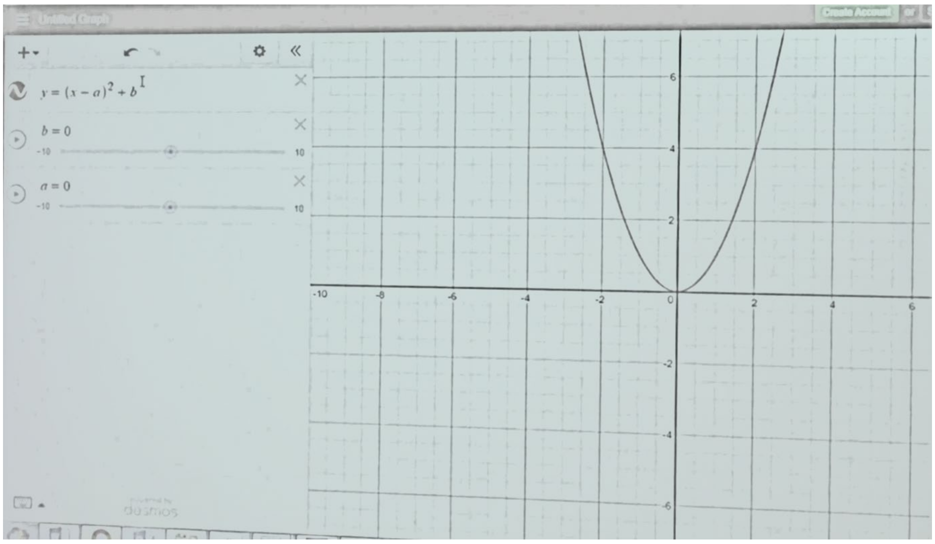
[Taken from Lesson 2 with a Year 10 class]

Returning to example 3.3a, as it can also demonstrate how students are given opportunities to engage cognitively with the mathematical ideas and relationships. In this example, students explore the features and relationships between the different methods for solving quadratic equations and the graphical representation of the quadratic equation they are working on. To this point, the students have been identifying the roots of quadratic equations by factorising the equations and making connections to the graph. The teacher then asks:

Teacher: Okay, so it is going to go through –2 and –5. So here solving helps us work out where our graph cuts the x-axis. Which leaves the question: What does completing the square do for us? How does that help us do anything?

The teacher then uses dynamic graphing software (see Figure 24) to vary the numbers of an equation in completed square form, $y = (x - a)^2 + b$. She begins by varying the number of the end (the constant term), the $+b$, and shows how the graph translates ("*if we change that, we get that movement in the* y *direction*"), by varying it across both positive and negative values of b. She then fixes the value of b to be 1 and varies the value of a and shows the curve translating left and right. She then fixes the value of a as 2.

Figure 24: Using dynamic graphing software to vary the numbers of an equation in completed square form (repeat of Figure 18)



She leaves the equation in the form $y = (x - 2)^2 + 1$ before asking:

Teacher: Now where does the 1 and the 2 come in if I were to write that out as an equation?

[She writes $y = (x - 2)^2 + 1$ on the whiteboard.]

Teacher: Now, so what do you think the significance of those numbers is? Because when we think about factorising quadratics, that helps us see where the line cuts the x-axis. So that is the significance of these two numbers for this graph. Can anyone spot anything there?

The teacher builds on this use of explicit patterns and generalisation to offer further opportunities for her students to engage in cognitively demanding subject matter. She goes on to vary the value of a, first to 3 and then to -1. At this point, the students identify the values as connecting to the *lowest point* or minimum point. The teacher then varies the value of a again to illustrate how the value of a relates to the x-coordinate of the turning point. The students have been analysing the different features of the equation in completed square form and the relationship to the graphical representation, and in doing so, they have determined the significance of the completed square representations. This task has thus enabled students to explore a range of features of and relationships between different representations, which go beyond recall or the rote application of procedures.

[Taken from Lesson 2 with a Year 10 class]

Another way in which some teachers helped their students to see the relationship between the turning point and the values in the completed square form of a quadratic equation was by beginning with the graph and using the line of symmetry to identify the turning point.

For example, for a graph with x-intercepts of -5 and 7, one teacher asks her students to identify the lowest point (minimum) of the graph (example 4.1d). The first student to answer identifies $x = 1$ as halfway between -5 and 7, and therefore as the line of symmetry for the parabola. Another student finds $x = 1$ by finding the average of -5 and 7. The teacher then asks for the y-coordinate, which a student conjectures to be -35. The teacher identifies this as the y-intercept and asks how students know it does not 'dip down a bit lower.' Another student suggests that they substitute $x = 1$ into the original quadratic equation, which results in the answer of -36. Here, different approaches have been taken to identify the minimum point of the graph and the students have used the relationships between the different representations and the features of the graphical representation in combination to develop different ways of identifying the minimum point.

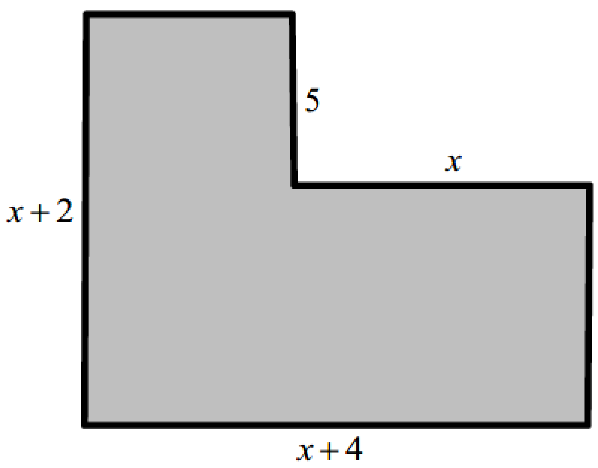
[Taken from Lesson 2 with a Year 10 class]

4.2 Task involving multiple approaches

In this final example (4.2a), students take different approaches to working on a problem in context. The students are given a compound rectilinear shape and the total area of the shape, and are asked to calculate the perimeter (as in Figure 25).

Figure 25: Finding the perimeter of a rectilinear shape when given the area

The shape shown has an area of 80cm^2. What is its perimeter?



To begin with, the students are given some time to think about the question and what method they will use to find the perimeter.

Teacher: Have a think about it, and think about what you know and what you don't know, and how you're going to tackle the question.

After a few minutes, the teacher then invites the students to share what they would do next:

Teacher: So, what is your gut instinct as to how to start the question? Sam?
Sam: Label the missing things.
Teacher: Label the missing sides? Are you going to work out an expression I can put on them? What are you going to go for then?
Sam: So, the top one would be $x + 4$ take away x.
Teacher: Can you tidy that up a little bit?
Sam: 4
Teacher: Okay, nice description though. Well explained. The top line is going to be $x + 4$ take away x so 4. What is this one going to be? Ashley?
Ashley: $x + 2 - 5$
Teacher: Which is?
Jo: $x - 3$
Teacher: Nice, well done. So $x + 2$ [points at left side], but I am subtracting the 5, gives me x take away 3.

The teacher then indicates that there are different ways in which the task could be approached, but without telling them what she would have done:

Teacher: Now what I find interesting is that you guys are not approaching the question as I did. Now that definitely is not a wrong thing and it is nice to know that

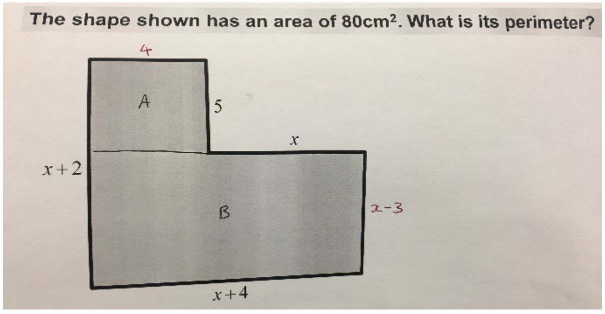
there are different approaches and all of them are valid, but you have got to think about how you would do it. It is a problem-solving question, so you don't have to do it the way that I teach you it, okay?
What do you think next? What other information have I got?

Taylor: The area?

Teacher: The area! So how am I going to use that?

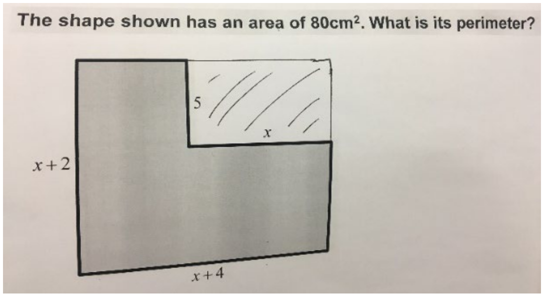
One student suggests splitting the shape into two rectangles as shown in Figure 26. The class first works out the expression for the area of shape A, and then the area of shape B, before forming an equation for the area of the whole shape.

Figure 26: Dividing the shape into two rectangles



Some other students have worked it out a different way, by adding in a missing rectangle, as illustrated in Figure 27, working out the area of the larger rectangle, and taking away the missing rectangle. This approach is shared with the class, who then discuss how this leads to the same equation, and consequently the same solution for the value of x.

Figure 27: Adding a missing rectangle



Here, the students have developed and shared different ways of solving a problem, working together with the teacher to determine that these different ways of finding the perimeter lead to the same solution.

[Taken from Lesson 1 with a Year 10 class]

4.3 Summary

The majority of videoed lessons in the TALIS Video Study included a task or activity which offered students opportunities to engage in cognitively demanding subject matter, such as analysing, creating or evaluating work that is cognitively rich and requires thoughtfulness. It was, however, rare for lessons to offer students more than one or two opportunities to engage in cognitively demanding subject matter. Classes with higher average attainment had more opportunities to engage in cognitively demanding subject matter than classes with lower average attainment, even though there are a range of cognitively demanding tasks and activities that are accessible to all students.[26] Classes that reported higher average levels of students' interest in mathematics also had more such opportunities than their counterparts with lower average levels of interest.

Mathematical ideas need to be thought about, reasoned with and discussed with others. Analysing these ideas and creating examples of quadratic equations for which a particular method would be more efficient than another, or where a particular method would not work, offers students opportunities to cognitively engage in mathematics.

[26] Watson, De Geest & Prestage, (2003)

5. Using multiple approaches to solve problems

There are several approaches to working with and solving quadratic equations. These include working with different representations of quadratic expressions, equations and functions, as well as different processes for solving the equations. The most common approaches for solving quadratic equations taught in classrooms in England include factorising, using the quadratic formula, completing the square or using graphical methods. Different approaches may be more efficient with some types of quadratic equations than others, but the different approaches also emphasise different properties of quadratic equations and quadratic functions which students can draw upon in their reasoning within broader areas of mathematics. The following case studies illustrate the different ways in which teachers in the study supported students to use multiple solution strategies and reasoning approaches.

The case studies involve students using two or more procedures or reasoning approaches to solve a problem or type of problem. The focus is on the processes involved in solving the problems, rather than the solutions to the problems. Importantly, it is the students who are using the multiple approaches, not just the teacher.

The examples of multiple approaches to and perspectives on reasoning are divided into three sections: using two methods for one question; all students working with multiple methods; and students choosing the method.

5.1 Using two methods for one question – and bringing them together

One way in which teachers can support students in using multiple approaches to solving quadratic equations is by explicitly asking the students to solve the same quadratic equation using two different approaches.

In this first example (5.1a), the students are split into pairs. One student in the pair is asked to solve the equation using factorisation and the other is asked to solve the equation using the quadratic formula, as shown in Figure 28. Several of the pairs are working on whiteboards that are on the walls of the classroom whilst the rest are working at their desks.

Figure 28: Tasks for solving quadratic equations using two different methods

Solve the following in pairs (one doing (a), one (b)):

$2x^2 + 5x + 2 = 0$

(a) by factorising **(b) by using the quadratic formula**

Discussion: Does it matter which method you use? When is it better to use the formula?

After the students have worked on this problem in their pairs, the teacher asks the students to check with their partner that they got the same answer. She then brings the class together, asking them whether they think they should get the same answer or different depending on the method used, before asking them to check whether they do indeed have the same answer as their partner. The class goes on to work through the question using the factorising method:

Finley: Times to get 4 and add to get 5.

Teacher: So, times them to get 4 and add them to get 5. So, what could those factors be? I have two numbers that when I multiply them together give me 4 but when I add them together give me 5. That could be...

Ari: 4 and 1.

Teacher: How could I rewrite this using the fact that I know you that you are going to factorise with 4 and 1. Drew, what did you do at this point?

Drew: $2x^2 + 4x + x + 2 = 0$ and then…

Teacher: Good. Does that make sense? So, we rewrote this with our $4x$ and our x, they are still going to add together and make $5x$, so we rewrote our equation with $4x$ and x, and then what do we do from here? Carry on then now Drew.

Drew: So, we split it into two parts, so you'd have your $2x^2 + 4x$ and you find the highest common factor.

Teacher: Lovely, the highest common factor of that would be…

Drew: $2x$

Teacher: And if I factorised it would be…

Drew: $x + 2$

Teacher: And then what would the second part be Fran? To factorise this bit? [Underlines $x + 2$ from the equation $2x^2 + 4x + x + 2 = 0$]. What would be the highest common factor of *x* and 2?

Fran: 1

Teacher: Absolutely right. So, I'd have my 1 here with $x + 2$.

[At this point, $2x(x + 2) + 1(x + 2)$ is written on the board.]
Teacher: So, then we can finish off. I have $2x + 1$ multiplied by $x + 2$ if I used the factorising method. And if I used the quadratic formula? So, I would be getting the same answer both ways? What would my two answers be? Bob, Sam, which answers did you get? x is equal to…
Sam: $x = -2$ and $x = -0.5$
Teacher: So, we have $x = -2$ and $x = -0.5$. Who else got the same answers?

In this example, the class has worked together through factorising the quadratic equation in order to find the two solutions. The teacher then asks students who have used the formula to find the solutions and what their final values for x are. The teacher asks students to raise their hands to show if they got the same solutions, before drawing the conclusion that the answers are the same whether students solved the equation by factorising or whether they solved it using the quadratic formula.

[Taken from Lesson 1 with a Year 10 class]

In this second example (5.1b), the students are again split into pairs and are working on the same quadratic equations, but with one person solving each equation and the other sketching a graph of the equation. The class work on the task (see Figure 29) for around 20 minutes before the teacher invites three students up to the board to share their sketches of the graphs for each of the equations.

Figure 29: Solving quadratic equations in pairs

WALT: use the discriminant

Person A: solve each of the following equations
Person B: Find the turning point and y-intercept and sketch the corresponding graph.

1) $x^2 - 6x + 9 = 0$

2) $x^2 - 4x - 2 = 0$

3) $x^2 + 2x + 2 = 0$

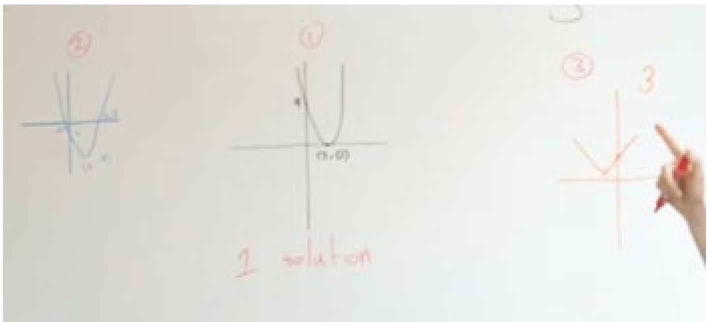
Compare with your neighbour. What do you notice?

The class comes together to talk through the three graphs that have been sketched by half the class (shown in Figure 30). The teacher then leads a discussion about the similarities and differences between the three graphs:

Teacher: We know what is the same about these three graphs. We know they are all parabolas, so they have all got a quadratic function.

What I am interested in is, what is different about those graphs? Can you tell the person next to you, what is different about those three graphs, how are they different?

Figure 30: Three sketches of quadratic functions



Initially, the students discuss the differences in pairs before the teacher collects suggestions from the whole class. The discussion initially focuses solely on the sketches, including the differences between the turning points and y-intercepts of the graphs:

Teacher: Anybody else have anything that is different? Jessie?
Jessie: Whereabout they turn.
Teacher: So, they are in different places and we got the idea that the turning point is in a different spot.

After a short period of time, the teacher shifts the attention of the class to the equation forms:

Teacher: When you did the first equation, Henley, how many solutions did you get for that first equation? How many different answers were there? So, this one had...
Henley: One.
Teacher: One solution. When you did the second equation, Sage or Henley, I am not sure which one you were doing, how many solutions did you get for the second equation?
Henley: Two.
Teacher: So, this one had two solutions. And lots of people had a problem with this last one. Blair, what happened when you tried this last one?
Blair: There was no answer, because …
Teacher: How can I look at the graph? How does the graph connect up to how many solutions they've have got? Again, person next to you, a few seconds.

Again, the students are invited to discuss the connections between the graphs and the number of solutions in pairs before they work on it as a class.

Teacher: Okay, alright, so this is the other thing that is different about these three graphs. Reese or Skylar, how does the graph connect up to how many solutions I have got? What is the connection?
Skylar: How many points there are …

Teacher: How many points there are. What do you mean, how many points there are? So, there is a point here, there is a point here, there is a point here, there are infinite points. [Points at different parts of the graph on the board.]
Skylar: No, like, where they cross the y-axis.
Teacher: Cross the... So, this one crosses the y-axis there [points at graph 2], this crosses it there [points at graph 1], that crosses the y-axis there [points at graph 3]. Is that what you mean?
Skylar: Not sure.
Teacher: Okay, somebody I haven't heard from yet, really. Spencer.
Spencer: How many intercepts there are.
Teacher: How many. What type of intercepts?
Spencer: x
Teacher: How many x-intercepts. Skylar, is that what you meant?
Skylar: Yes.
Teacher: Okay. Remember that language. x-intercepts. That got two [points at graph 2], that got one [points at graph 1], and that got none [points at graph 3].
Teacher: So, the number of solutions an equation has tells us about the graph. And some equations have no solution. Look at that graph of that last one, that has no solution and some people were a bit bothered by that on their calculators.

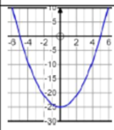
The lesson moves on and the teacher describes what the discriminant is and how this connects to the quadratic formula. Subsequently the class work in pairs to discuss why, by just looking at the quadratic formula, they can deduce whether there can be two solutions, one repeated solution, or no real solutions.

[Taken from Lesson 2 with a Year 10 class]

5.2 All students working with multiple methods – and bringing them together

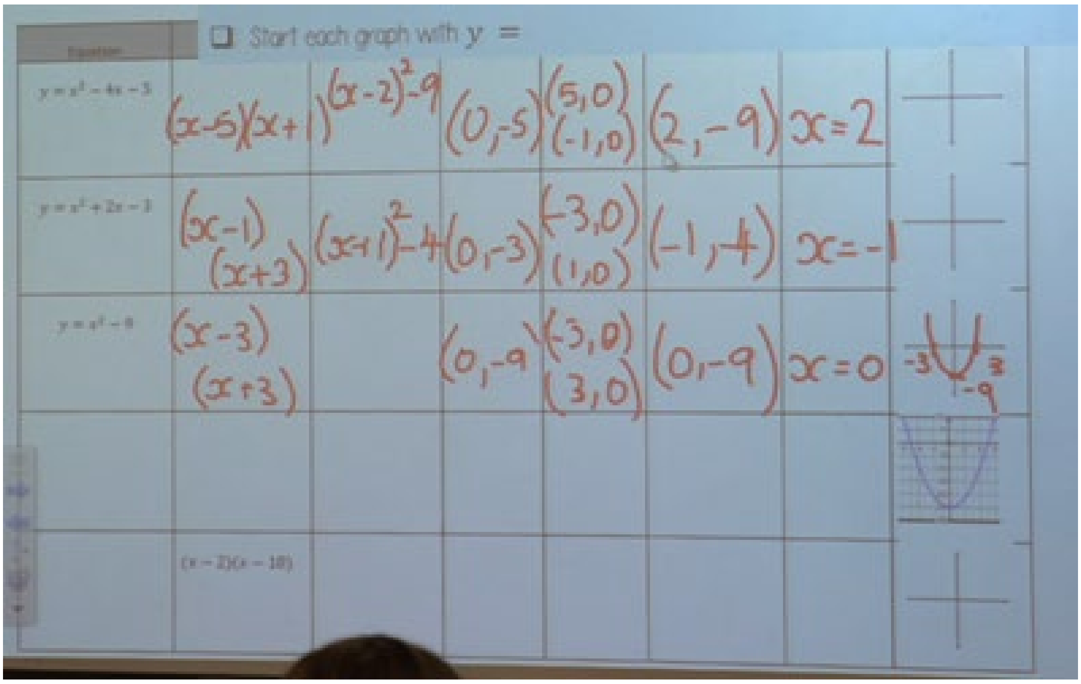
In this next example (5.2a), students were asked to complete a task in which they were given three equations, for which they had to factorise, complete the square, find the y-intercept, the roots of the x-intercept, the turning point and line of symmetry, and then sketch a graph (as shown in Figure 31).

Figure 31: Task bringing together multiple methods

Equation	Factorisation	Completing the square	y-intercept	Roots x-intercept	Turning point	Line of symmetry	Sketch
$y = x^2 - 4x - 5$							
$y = x^2 + 2x - 3$							
$y = x^2 - 9$							
							
	$(x - 2)(x - 10)$						

Initially, the teacher asks the students to complete the first three rows of the table where the equation is given. After some time working on this individually, the class come together to complete the rows on the interactive whiteboard (except for the graph in the final column) for the first three equations (see Figure 32).

Figure 32: The table in Figure 31 partially completed on the whiteboard



The teacher then uses this final column as a way of bringing together all of the information the students have worked out in the other columns of the table, drawing from different methods and approaches that can be used to find out the different representations, coordinates or values.

Teacher: Once you've completed those three lines, I want you to try and find how I can find the turning point. I've not graphed them, I've not put the sketches on there. Try and find a connection between the turning point and something else on the line.

The teacher gives the students some time to think about this individually, before adding:

Teacher: I'm not drawing the sketches; I don't need them to find the turning point
.

The students are then asked to discuss how to find the turning points in pairs. Finally, the teacher asks students to share their thoughts in a whole class discussion:

Teacher: Keep your hand up if you've got an idea, how I can find the turning point?
Kelly: Completing the square.
Teacher: You're on the right lines, this is why we complete the square.
Kelly: Um…
Teacher: It's something to do with when we complete the square. Francis?
Francis: Is it like the number you square but the opposite of it.
Teacher: Yes, because think about that minimum value, I know that the lowest number here [circling $(x-2)^2$], I need that bracket to be zero. Zero squared is going to give me zero. It's going to give me the smallest number.

The teacher goes on to explain the relationship between the expressions in the completed square form of the equation and the x and y coordinates of the turning point, before bringing all the information together to sketch the graphs in the final column.

[Taken from Lesson 2 with a Year 10 class]

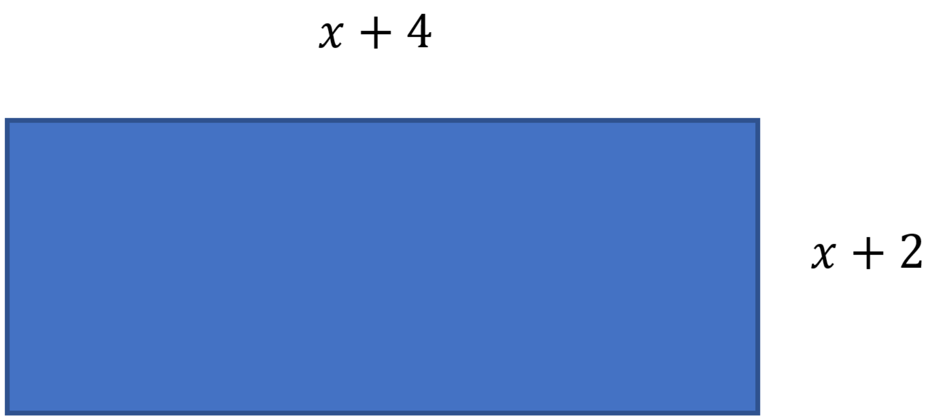
While the discussion shared above here for this task focused on which method is most useful for identifying the coordinates of a turning point, the students also factorised to find the roots of the equations, and have used the completed square form to identify the line of symmetry. Tasks like these offer students and teachers the opportunity to see when and why particular approaches to solving or working with quadratic equations might be used.

5.3 Students choosing the method

Another way in which teachers offer students the opportunity to use two or more procedures or reasoning approaches, is by providing them with questions where they can choose which method to use. This can be particularly useful where there is the opportunity to discuss or deduce why one particular method for a particular type of quadratic equation might be used, or why one method may be more efficient than another.

In this next example (5.3a), the class has been working on finding the area of rectangles where the sides are given in algebraic expressions, as in Figure 33.

Figure 33: Finding the area of a rectangle



The students have been given several rectangles like this, from which they have been constructing an algebraic expression to represent the area of the rectangle. The teacher has not specified how students should find the area. Some students have broken the rectangle down into smaller rectangles, others have worked solely with the algebraic expressions. Some students have solutions which are quadratic expressions in expanded form, whilst others have them in factorised form. Towards the end of the lesson, the teacher asks:

Teacher: Do you think, that if we have to do this for exams, let's say, is this an efficient way to do it? To draw rectangles?
Students: Yes … no.
Teacher: Why not, Alex?
Alex: Because it is going to take a really long time.
Teacher: Excellent. So, it is a good way, but it is not the most efficient way. Because we will not have squared paper with us, we can't do it.

The teacher then goes on to ask the students to "*try and find the relationships*" between the numbers in the linear expressions that denote the lengths of the sides of the rectangles, and the numbers in the quadratic equation that the students have found as representing the area of the same rectangles.

[Taken from Lesson 1 with a Year 10 class]

5.4 Summary

In England, the majority of the teachers and students participating in the TALIS Video Study said that teachers frequently compared different ways of solving problems, but only around 10% of the lessons videoed also included opportunities for students to use two or more procedures or reasoning approaches to solve a particular problem or type of problem. In the majority of lessons, students generally used a single procedure or reasoning approach at a time. Classes with higher average prior attainment experienced more opportunities to use two or more procedures or reasoning approaches compared to classes with lower average prior attainment.

To be fluent in solving quadratic equations, students need to know *when* to use a particular process or procedure. This involves understanding the relationships and connections between the different solution strategies,[27] as well as recognising the different features of quadratic equations.[28] Using more than one means of representing concepts in teaching draws attention to critical aspects, which can contribute to a deep and holistic understanding of those concepts. By paying attention to what is kept the same and what changes - whether that is comparing different representations or different solution methods – the teacher draws attention to mathematical relationships and structure. Students should be supported to compare different methods and to make choices about which method or strategy to use. Recognising the different features of quadratic equations is an important part of identifying which methods to use to find solutions in effective ways. Similarly, students can be supported to appreciate the underlying structure of the equations by helping them to recognise that all quadratic graphs can be transformed to look like $\pm x^2$ by translating and scaling, alongside developing an understanding of what the different terms in different representations of a quadratic equation can tell us about these graphs (and about the translating and scaling that has taken place).[29]

[27] Nunes et al., (2009)
[28] Block (2015)
[29] Mason et al. (2009)

6. Students engaging in opportunities to understand the rationale for processes and procedures

Understanding of subject matter procedures and processes describes activities where students engage in opportunities to understand the rationale for procedures or processes. This includes describing the goals or properties of a procedure, stating why a procedure or a solution is the way it is, or visually designating the elements or steps in a process or procedure. For example, students might explain why they would use a factorisation approach for a particular problem, rather than completing the square. Students might also compare and contrast the different methods for solving a quadratic equation to identify where one method might be more advantageous than another. Other activities could involve students identifying why a particular method might not work for the problem they are currently working on. Most importantly, it is the students that are engaging in understanding the rationale, rather than the teacher explaining or describing it.

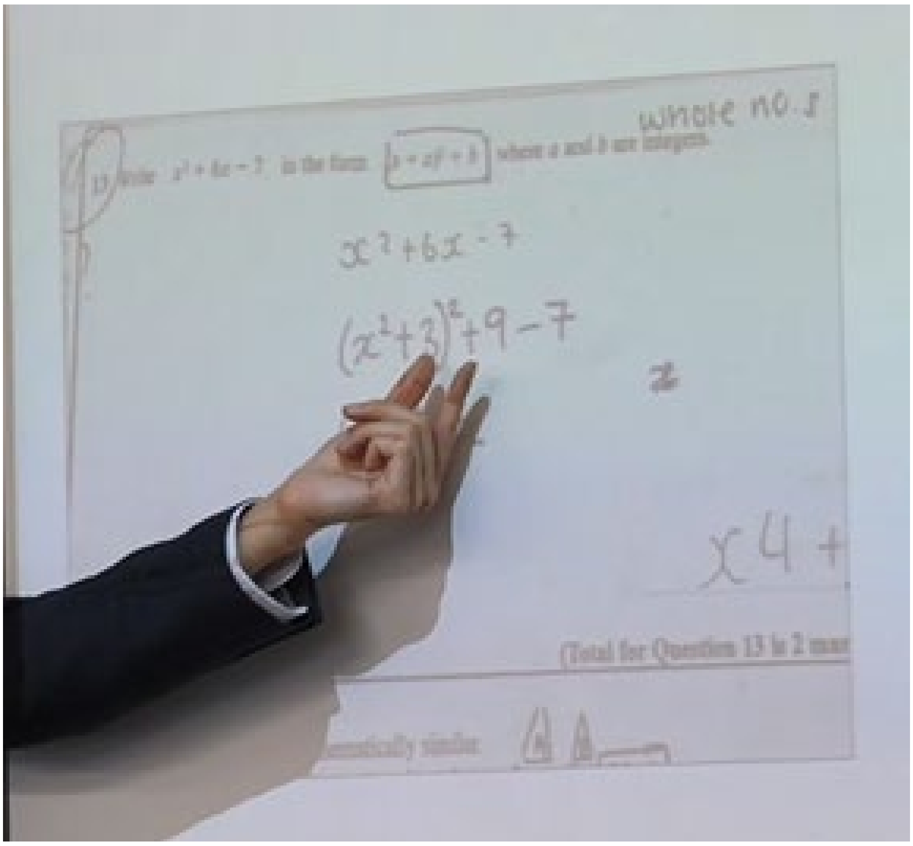
In this section, the examples of understanding of subject matter procedures and processes are grouped into three sections: identifying errors and explaining the issue; visually designating the elements or steps in a process or procedure; and students asking questions.

6.1 Identifying errors

In this example (6.1a), the teacher has projected a solution to an equation on the whiteboard (see Figure 34). The solution contains an error, and the teacher asks the students to identify the error that has been made:

Teacher: If any of you want a challenge, try and see if you can spot the mistake that this Year 11 student has made. [Points at board.] What has he done wrong?
Sawyer: He swapped it around. So, it should have been add 7 and not take 7.
Teacher: Yeah, no, so he still has to take 7 because that was the thing, but you are saying that he added 9. What should he have done?
Sawyer: Take away 9.
Teacher: Take away 9, yeah. Because the 9 is that extra bit that we need to complete the square on to make it right. So, he should have taken 9 away.

Figure 34: Screenshot for an example where a student has made an error



The students then continue to work through the problem once the $+9$ has been changed to -9. The teacher then directs the students to identify another mistake within the same solution:

Teacher: What is his other mistake?
Blake: He put the square in the bracket.
Teacher: He's put the square in here but where should it be?
Students: Outside.
Teacher: Exactly, it should only be on the outside. So as a teacher, what you should be able to spot there is he almost got the idea, he has the basic concept somewhere in his head, but he has a lot of misconceptions going on. And if you can spot misconceptions from other people, it means you fully understand it.

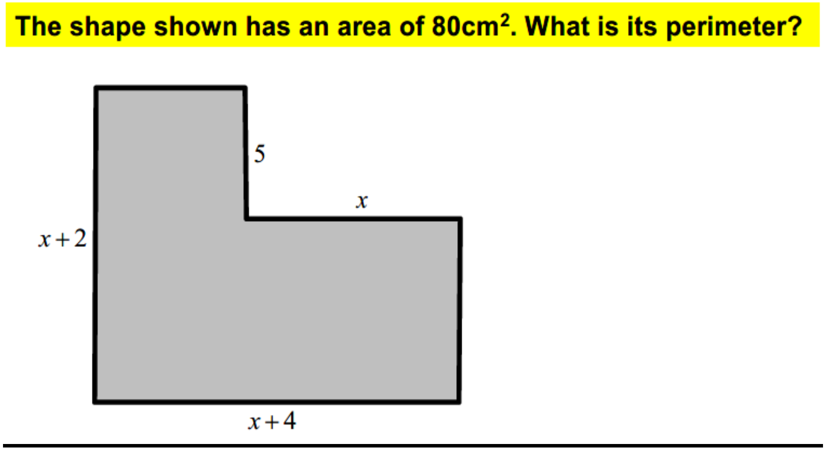
In this example, the students have identified two errors and identified what the corrections would be. For the first error, the students gave a brief explanation for why the error may have occurred. The explanations offered by the teacher and the students focus on the procedure for completing the square and how the mistakes have arisen from deviations from this procedure.

[Taken from Lesson 1 with a Year 10 class]

6.2 Visually designating the elements or steps in a process or procedure

This next example (6.2a) returns to an earlier example (4.2a) in which students have been asked to find the perimeter of a compound rectilinear shape given only some measurements and the area of the shape. As well as enabling the students to engage in cognitively demanding subject matter, the students are also making connections between visual representation and the steps being taken to solve the problem.

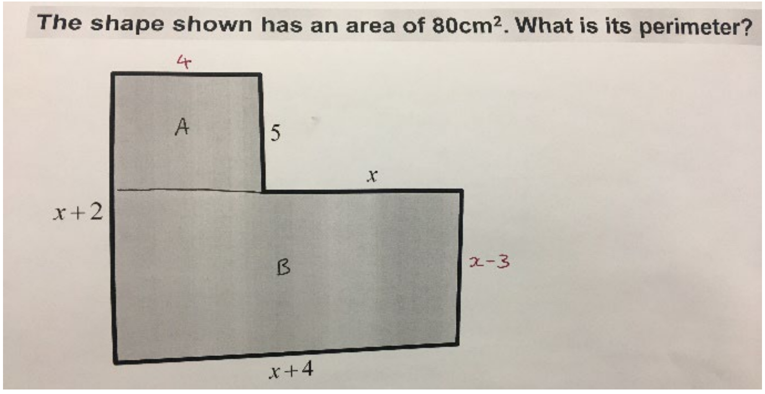
Figure 35: Finding the perimeter of a rectilinear shape when given the area (see also Figure 25)



At this point, the class have labelled the two missing side lengths in terms of x.

Teacher: What do you think next? What other information have I got?
Taylor: The area?
Teacher: The area! So how am I going to use that? I heard a light bulb in Hayden's head. Go on Hayden.
Hayden: Work out the area of the square and then work out the area of the rectangle. Well, it is not a square, it's also a rectangle.
Teacher: Well corrected! So you want to split it? There? So, Hayden is going to split it there because it is a compound shape. We don't know of a formula to do the area of that shape, there isn't a standard one. So, we are going to split it into shapes we know. [See Figure 36.]

Figure 36: Dividing the shape into two rectangles (see also Figure 26)



Teacher: We have a little rectangle on the top, which has an area of …?

Rowan: 20 cm^2.

Teacher: 20 cm^2. I am going to label them A and B just because it is easier for the way I marked it down. So, area for A, we are saying is $4 \times 5 = 20cm^2$, okay? Please make sure you put your units on.

Teacher: How am I going to do the area of part B? Alexis?

Alexis: You do the brackets, $(x-3)(x+4)$.

Teacher: Why? Why did you times those two things together?

Alexis: Because that is the height and the width.

Teacher: Yes, thank you. I was just making sure that everyone knew why you used $x - 3$ and not $x+2$ [points at left side]. Right? So, I am using that rectangle at the bottom, which has a height of $x-3$ and a width of $x+4$. So, they are my two measurements.

The students calculate the area of the two rectangles and then, through the teacher's questioning, the connection between the expressions and the lengths of the sides is made explicit. The students continue to work through the problem until they reach the point where they have an equation for the area of the compound shape. They then work through simplifying the equation into a form that they can factorise:

Teacher: Okay, so my total area is $x^2 + x - 12 + 20$ and I know that is going to be equal to 80.
So, at this point, I am not putting my units on because they are both in terms of the same unit and I am interested in solving to find x. When we get to the end of the question, we are expected to have that unit in, but for now, while we are solving the equation, it actually makes it more challenging if you leave it on both sides.
Right, so what am I going to do now? Charlie?

Charlie: Could you do $x^2 + x + 8 = 80$?

Teacher: $x^2 + x + 8 = 80$, did you say?

Charlie: Yes.

Teacher: Yeah, really good, well done. Anyone see what is going to happen next? Emerson?

Emerson: Can I factorise it?
Teacher: Can I factorise it now?
Emerson: Ah no. We have to do – 80.
Teacher: Why?
Emerson: So it is equal 0.
Teacher: Yeah, well done. I need it to be $= 0$ because the process we know. We need the two things that multiply together to be 0 and we know one of them is 0. So, I am going to subtract 80 from both sides. What will I get?
Emerson: $x^2 + x - 72$
Teacher: Take away 72, okay. Can you try and factorise that for me and solve it like you have been just doing? So you can do that last bit?

The students then work individually while the teacher circulates, until the teacher brings the class together and asks a student to share their solution:

Teacher: Have you got it, Morgan? Go on then please.
Morgan: $(x + 9)(x - 8)$
Teacher: Very nice. What would x be?
Student: -9 or $+8$.
Teacher: $x = -9$ or $x = +8$. Can both of these things be the answer? Elliot, what do you think?
Elliot: Only one of them.
Teacher: Why?
Elliot: Because the shape cannot come from a negative.
Teacher: If x is negative 9, how long is this bit of the line?
Elliot: -9.
Teacher: Can you have a length of -9? Right? So, we know for this particular question, although there are two solutions to the equation, there is only one solution to the problem. Alright?

Here, the teacher has prompted the student to explain why the solution to quadratic equation they formed together as a class does not work for the problem they are currently working on. In this situation, the context of the problem, perimeter and lengths of sides, means that one of the solutions, $x = -9$ needs to be ignored as lengths cannot have a negative length. In this lesson, the students have generated an equation that represents the area of the shape, before using this equation to calculate the lengths of the different sides and subsequently to calculate the perimeter of the shape. Throughout the lesson, the connections between the visual representation of the shape and the algebraic representations have been made by both the teacher and the students. Many of these examples could also illustrate the use of explicit connections in the teaching of mathematics.

[Taken from Lesson 1 with a Year 10 class]

6.3 Students asking questions

Another way in which teachers can check that students are engaging in understanding the mathematics involved in solving quadratic equations is through attending to the questions that they ask. In this example (6.3a), the students are asked to answer the problem in Figure 37.

Figure 37: Task involving factorisation into completed square form

Express $2 - 4x - 2x^2$ in the form $a - b(x + c)^2$

The teacher talks the class through the manipulations, expressing it as $-2x^2 - 4x + 2$ and then as $-2(x^2 + 2x - 1)$, before asking the students to complete the square on their own. The teacher, in turn, shows that $-2[(x + 1)^2 - 1 - 1] = -2[(x + 1)^2 - 2] = -2(x + 1)^2 + 4$.

At this point, a student asks a question about the mathematics:

Jaylen: Where did the extra -1 come from?
Teacher: From the coefficient of the middle term. So, it becomes -1 outside, so if you have a -1 and a -1, it becomes -2. Half the coefficient of that.
Jaylen: No, no, how did you go from that equation, 1 –1, to the next one with the 2?
Teacher: The 2? Because you have -1 and -1.
Jaylen: Yes, I know that but where did the other -1 come from?
Teacher: Ah because remember…
Jaylen: Ah… because it was another -1 squared?
Teacher: We have 1 squared, when you take away -1 squared.
Jaylen: Ah and then the existing -1 and after you opened that up you have another -1.
Teacher: Yeah and you get -2. So, you have that -1 and that -1, which is equal to -2. Yeah? Do you get it?
Jaylen: Yes.
Teacher: Okay.

Here, the teacher takes the time to answer the student's question and to persevere with this until the student is happy that their question has been answered. In this example, the fact that value -1 appears in more than one place has created a sense of ambiguity, but through a combination of the teacher's support and the student being given the time to re-ask their question, the student comes to understand where each of the terms has come from.

[Taken from Lesson 2 with a Year 9 class]

In this next example (6.3b), the class have been working on $x^2 + 5x - 4 = 32$ and rearranging this equation into completing the square form. The class has talked through subtracting 32 first, then halving the coefficient of the x term to get 2.5, resulting in the expression $(x + 2.5)^2$. They then calculate 2.5 squared to get 6.25, which they then subtract from 36 to get −42.25.

Teacher: Yeah, Ari?
Ari: Could you put 36 as 6 squared?
Teacher: Yes
Ari: And then work it out like that?
Teacher: What do you mean? Would you do 2.5 take away 6 and then square it?
Ari: Yeah.
Teacher: Ah no, it doesn't quite work like that.

The student's question here focuses on the step of the solution that moves from $x^2 + 5x - 36 = 0$ to $(x + 2.5)^2 - 6.25 - 36 = 0$. This would involve expressing $x^2 + 5x - 36 = 0$ as $x^2 + 5x - 6^2 = 0$, which is an equivalent equation, but then treating the expression $(x + 2.5)^2 - 6^2$ as equivalent to $(x + 2.5 - 6)^2$. This is a common error that arises when students treat squaring as linear.

[Taken from Lesson 1 with a Year 9 class]

6.4 Summary

The majority of videoed lessons in England included some activities in which students engaged in opportunities to understand the rationale for procedures or processes. Students would ask questions about why a procedure is the way it is, or would describe the goals or properties of these procedures and processes. Yet for much of the time, students engaged with procedures or processes without attending to the rationale behind them. Students were given plenty of opportunities in their lessons to practice procedures and different methods for solving quadratic equations, and plenty of opportunities to develop their fluency with individual methods. What was less common was giving them the opportunity to consider why a procedure is the way it is, or why they might choose one method over another. Classes with higher prior attainment experienced more opportunities to understand the rationale for processes or procedures than classes with lower prior attainment.

Students can develop their understanding of mathematics by being prompted to identify and explain errors; relating the steps in a process or procedure to a visualisation or representation, or by asking questions. Knowing how to solve a quadratic equation fluently is important, but it is not sufficient. Students also need to understand when and why a particular solution strategy works. Analysing examples of common errors or misconceptions is one way of supporting students to understand why a particular solution strategy is the way it is. Asking students to make the connections between a visualisation or representation and a process or procedure is another way.

7. Teacher and student explanations

Explaining is a common practice in mathematics classrooms. Explanations can come from the teacher or the students, or can be co-constructed by teachers and students, and can be used with the whole class, small groups or with individuals. The explanations from the study which are explored in this chapter include descriptions of why ideas or processes are the way they are. They were defined as statements that clarify, rationalise or justify mathematical ideas, procedures or processes.

This section illustrates two types of explanations of why ideas or processes are the way they are. The first of these considers explanations given by the teachers in the study. The second focuses on those explanations given by the students.

7.1 Teacher explanations

Explanations can vary considerably in length but explaining why something is the way it is does not necessarily have to take much time.

The first example below (7.1a) involves a quick explanation of why, when solving the equation $x^2 - 16 = 0$, it is important that one side of the equation is equal to zero. The teacher does this by making connections to the graph of the quadratic function.

Teacher: So, If I am solving it, I want it to be equal to 0, because the solution is where it crosses the x-axis, and the x-axis is when y is equal to 0.

[Taken from Lesson 1 with a Year 10 class]

In the second example (7.1b), the students have been working through a sequence of equations starting with equations like $7x = 0$, before building to equations like $5(x + 1) = 0$, and then finally to quadratic equations in the form $(x + 2)(x - 3) = 0$. They are doing so in order to focus on the zero product property (if $a \times b = 0$, then either $a = 0$ or $b = 0$, or both). The teacher summarises the discussions as follows:

Teacher: Because we said before that two things multiplied to make 0, then one of them must be 0. So, either this 5 times, exactly like George said, the brackets, so either the 5 is 0, which it's not, or the brackets are 0. And if the brackets are 0 then $(x + 1)$ is 0, and exactly like Jordan has just said, that means x is -1.

Here, the teacher has drawn upon the students' earlier brief explanations for different tasks and combined them in this explanation.

[Taken from Lesson 1 with a Year 10 class]

7.2 Students explaining reasoning

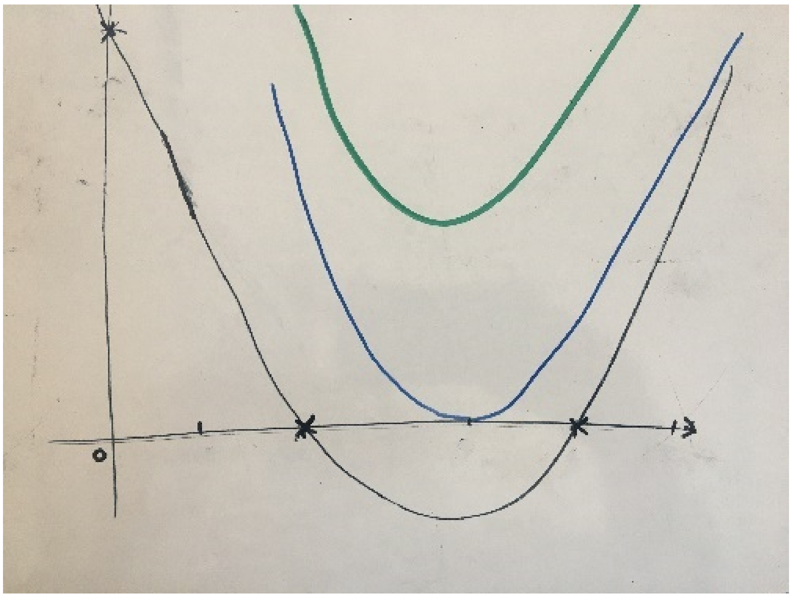
Students can be invited by their teachers to offer explanations using the prompt ‘why?’. In this first example (7.2a), the lesson is focusing on getting from a quadratic equation “to knowing what the two values possibly are that make the function equal to zero, that solve this equation”. The teacher begins by asking:

Teacher: Will there always be two? Who thinks yes?
Students: [some raise their hands]
Teacher: Who thinks no? [no hands raised]
Teacher: Who is not sure?
Students: [some raise their hands]

The interaction then continues with the teacher asking some students to explain why they think there will always be two values, or not.

Teacher: Why do people who think that it is always going to be two, think it has to have two answers? Don’t let Sam do everything, but go on, Sam, you do this one.
Sam: Because even when it ends up as [inaudible] and there’s just the square left, there will always be a $+$ and a $-$.
Teacher: So, Sam has remembered that when I have a square root, when I’ve got a squared number, can have a $+$ and a $-$ answer. Okay, nice, so there is two responses there. What about the graph? When you think about the graph, when I draw that sort of shape, will it always cross the x-axis twice? Ashley?
Ashley: No, because [inaudible] when it crosses the x-axis right at the bottom of the curve, there’d only be one.
Teacher: So, if my curve perhaps sat at the number 3, something like that [draws the blue graph in Figure 38 on the board]. Convinced? So two or one then? Is that it? Could it never touch it?
Nicky: Yeah.
Teacher: What would that mean, Nicky?
Nicky: That would mean like a [inaudible].
Teacher: Yeah, so actually, it could be up there and never touch it at all [draws the green graph in Figure 38 on the board]. So, there are three options. Even though you know that the maximum it can have is two, and a lot of them will have two solutions, some of them will have one and sometimes there won’t be any. So, don’t panic if you solve an equation and you find that there isn’t an answer. Because some of them don’t cross.

Figure 38: Three nested parabolas illustrating the different number of roots possible



In this short extract, there have been three explanations from students beginning with why there are usually two roots to a quadratic equation, followed by Ashley giving an example of when there might be one repeated root, and Nicky explaining why there could be no roots. Each of the students' justifications for why there could be just one repeated root or why there could be no roots includes a general example of a quadratic curve, like the one given to illustrate the case where there are two roots, which the teacher draws on the board for the whole class to see.

[Taken from Lesson 1 with a Year 10 class]

Another way in which teachers prompt students to offer explanations is to invite them to explain why a problem or equation does not fall within a particular category. In this next case (7.2b), the class have been working on recognising different arrangements of quadratic equations, as well as different cases that depend upon the coefficients within the equation. The task shown in Figure 39 involves students identifying which of the equations are perfect squares, and which are the differences of two squares.

Figure 39: Task designed for students to identify perfect squares of the difference of two squares

Level 2 – perfect squares or difference of squares

1. $x^2+6x+9=0$	4. $x^2-49=0$	7. $x^2-x+\frac{1}{4}=0$
2. $x^2-16=0$	5. $x^2-20x+100=0$	8. $x^2-13=0$
3. $x^2-10x+25=0$	6. $x^2+2x+1=0$	9. $x^2+4=0$

Teacher: Which ones of these are perfect square questions, can you tell? Go on then, Alex.
Alex: The ones where the front numbers have a squared number in them.
T: So, this one because it has a square number [circles example: $x^2+6x+9=0$]. This has got a square number as well though [points at $x^2-49=0$], so is that a perfect square?
Alex: No.
Teacher: Why not?
Alex: Because it hasn't got like a $+6$ …
Teacher: Right, it hasn't got that x-term, coefficient, number of x. Alright, so what I need is this number here, Alex said is a square number, and how does it link to this one [the x-coefficient] then?
Alex: The square root is 3 and 3 + 3 is 6.

They then go on to check this relationship between the constant term and the x-coefficient across more of the examples on the whiteboard, before generalising to quadratic equations.

[Taken from Lesson 1 with a Year 10 class]

7.3 Summary

In the TALIS Video Study, around three quarters of the participating teachers and students reported that there were frequent opportunities for students to explain their ideas in class. The majority of lesson videos also included explanations of why ideas or processes are the way they are, though in many cases, these explanations were brief or focused on superficial features of the mathematics. Higher attaining classes were more likely to experience lengthier explanations focused on deeper features of the mathematics compared to classes with lower average attainment.

Explanations are one of the most prevalent discursive practices in mathematics classrooms. Learning mathematics is about knowing ‘why’ as well as knowing ‘that’ and knowing ‘how’. Explanations by teachers and students that focus on ‘why’ support students in understanding the mathematics they are learning. Student explanations not only demonstrate what they understand but can also help them to clarify their own thinking and become aware of misunderstandings or a lack of understanding.[30]

[30] Ingram et al. (2019)

8. Teacher feedback through conversations

Feedback has been shown to have large positive effects on learning in mathematics[31] and there are a variety of ways in which teachers can give their students feedback. Verbal feedback in particular can be given immediately after a student answers a question, when they have identified something they do not understand or when they ask a question. This section focuses on feedback that occurred as part of a conversation between a teacher and a student or students – that is, in feedback loops rather than feedback in the form of a single response from the teacher.

Examples in the TALIS Video Study highlight the teacher's responses to students' thinking, and focus on particular interactions between the teacher and a student or students, which examine why the students' thinking is correct or incorrect, or why ideas or procedures are the way they are. The mathematics discussed by the teacher and student also needs to be addressed in a detailed or complete way.

This definition of teacher feedback relates to three distinct situations: where students make a mistake or reveal that they have a misconception, which the teacher then responds to; where students have correctly answered a question and the teacher responds by exploring why their thinking is correct; and where a student asks a question that suggests that they do not understand something or are not following.

8.1 Working with individual students

One common situation in which teachers respond to student thinking is when the students are working independently and request help. In this first example (8.1a), the teacher is walking around the classroom checking on individual students and stops to work with one particular student who appears to be struggling.

Teacher: Alright, where did this 5 come from?
Alex: I don't know.
Teacher: Okay, so what are you trying to make it equal to?
Alex: 0
Teacher: Right, I reckon, how am I going to eliminate this?
Alex: Minus 9.
Teacher: Alright, so if I take 9 away from here, what is left?
Alex: 6
Teacher: Okay, So I reckon if...
Alex: I'm not sure.
Teacher: No, I'm not sure either. If I subtract 9 from both sides, that is what I am going to have, right?

[31] Hodgen et al. (2018)

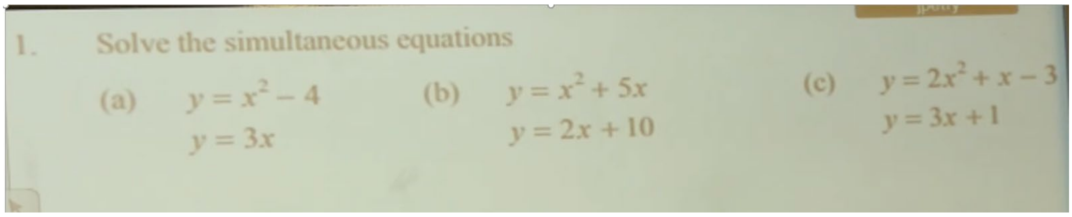
Alex: Yeah.
Teacher: Now, look at the equation. What do you notice about all the numbers?
Alex: 3 goes into them all.
Teacher: 3 goes into them all. So you can either, bring 3 out as a factor, because as you said, it goes into them all, you mean it is a factor of everything, so you could bring it out front. Or there is something else you could do, which might make the question a bit more straightforward. […] If you want to divide everything on the left-hand side by 3.
Alex: Send this over.
Teacher: Yeah, actually you would be dividing anyways because I want to do the same to both sides but it would still be 0, so I could actually share the whole equation by 3, make my numbers a bit smaller, which actually ultimately is probably more straight forward than bringing it out in the front. Because it is equal to something you can do it to both and know it is going to match, right?

The teacher has supported Alex in working step-by-step through the procedure for solving a quadratic equation where the equation is being rearranged before being solved. The interaction ends with the teacher sharing two possible strategies for dealing with the fact that 3 is a common factor of each term in the equation, before justifying why one strategy may be better than the other – or at least more straightforward, in this case. Whilst the student's method of dividing both sides by 3 is appropriate and correct, the teacher has introduced a different strategy in their feedback.

[Taken from Lesson 1 with a Year 10]

In this second example (8.1b), the students are again working individually, while the teacher circles the classroom supporting students in turn. The following discussion focuses on the first of the three questions shown in Figure 40.

Figure 40: Task involving solving simultaneous equations


1. Solve the simultaneous equations

(a) $y = x^2 - 4$, $y = 3x$ (b) $y = x^2 + 5x$, $y = 2x + 10$ (c) $y = 2x^2 + x - 3$, $y = 3x + 1$

Teacher: So, in this step here, you are trying to make it all equal to 0. So, to make it all equal to 0, you have to take this to the other side, so you've got to take away $3x$. Do you want to change that to take away $3x$?
Sara: Yeah.
Teacher: So, you take away $3x$, you can't just take away that x by the way, to make it just 3. You've got to take away $3x$ all in one. So, you get $x^2 - 3x - 4$. Yeah, you've got to put the $-3x$ on this side now. So, you get $x^2 - 3x - 4$ and 0 on this side.
Sara: Could you repeat that please?
Teacher: So, to get this equal to zero, you have to take this $3x$ away.

Sara: Oh yeah, that makes sense.
Teacher: Yeah, you can't just take away the x, because it is actually a multiplication. It would go straight to 0 if you took away the $3x$.
Sara: So, is this step not [inaudible] and I could have used it anyway?
Teacher: No, this is incorrect because you can't take away the x from $3x$. This is a multiplication. The only way you could get rid of the x in $3x$ is to divide away the x and that creates more problems than it solves really. It doesn't simplify the equation.
So, if you take away $3x$ on both sides that is the way to make it equal to 0, but you end up with $x^2 - 4 - 3x$. I'll write it out for you. So, you end up with $x^2 - 3x - 4$. So that is what happens if you take away $3x$ from both sides and then you see if that factorises. Because now it is equal to 0.

Again, the teacher is supporting the student to work step-by-step through the question. Initially, the teacher states that the student can't just take away the x, but has to take away the $3x$, the whole term. The second time that this is stated, the teacher gives a reason why, "*because it is actually a multiplication*". When the student questions this, the teacher repeats the reason before expanding the explanation further to include what could be done, dividing by x, before explaining why, in this particular question, this would not help.

[Taken from Lesson 2 with a Year 10 class.]

In the final example (8.1c), the teacher is supporting a student who is struggling to factorise a quadratic expression where the coefficient of the x^2 term is negative. In this feedback loop, the teacher leads the student through finding factors of 9 that sum to 8 but explains why in this particular case the pairs the student identifies will not work.

Teacher: You've gone for that one there. So, we've got a negative x^2, so what do we do there? What do you think we do?
Riley: That one is going to be $-x$ and that one just x.
Teacher: Yeah, a negative x multiplied by that x will give you that negative x^2. But you need to be careful because when you multiply negative x by that number here, you will get that. Do you see what I mean?
Riley: Yeah.
Teacher: So, then you think, okay, what two numbers will multiply together to get 9?
Riley: 3 and 3.
Teacher: 3 and 3? But add together to get 8?
Riley: 4 and 2.
Teacher: Add together, not multiply. So you need to think about two numbers that multiply together to make 9 and add together to make 8.
Riley: Oh, 9 and –1.
Teacher: Right, okay. But 9 times -1 is going to be -9. So how do you get around that? […] Hmm see, -9 times $-x$ will give you $9x$, okay? And then what?
Riley: Uhh.
Teacher: And then $+1$, if you had $+1$, then you would get $10x$, wouldn't you? Because that would be $9x$ and then add and x that would be $10x$. So, can you think of another way?

Riley: Hmm…
Teacher: I don't want to give you the answer because you need to think about it yourself.

The teacher leaves the student to work out for themself which pair of factors will work, and then the student goes on to complete the factorisation independently.

[Taken from Lesson 1 with a Year 10 class]

8.2 Summary

The majority of lessons from England included interactions between the teacher and the students which focused on why students' thinking is correct or incorrect, or why ideas or procedures are the way they are. What was rarer were interactions that addressed the mathematics in a detailed and more complete manner.

As highlighted above, learning mathematics is about knowing 'why', as well as knowing 'that' and knowing 'how'. The feedback loops described above emphasise why something works, as well as why alternative potential strategies either do not work, or may not be the most appropriate strategy in a given situation. As teachers' practices shift towards providing more verbal feedback that focuses on students' learning,[32] examples like the loops considered in this chapter will become more common. Giving students specific feedback on why their thinking is correct or incorrect is known to be an effective means of supporting students' learning,[33] and feedback loops are a means of doing this in a way that is responsive to students' thinking.

[32] Elliott et al. (2020)
[33] Higgins et al., (2016)

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